SwissCube Control Algorithm Design and Validation

Master Project

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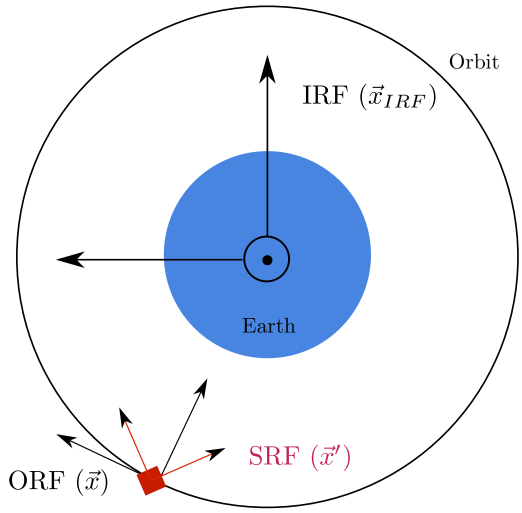
1 Introduction

The SwissCube is a student built nano-sattelite in the CubeSat format. Its main characteristics are a mass of $1kg$, and a volume of $10cm \times 10cm \times 10cm$. These very small dimensions make the design of a full featured satellite very challenging. This document is the report for a Master project and approaches the problem of the attitude control of the SwissCube. The goal is to make a theoretical study of the satellite's dynamics using analytical mechanics and of the control possibilities given by actuators at disposition.
To influence its attitude (i.e. its orientation relative to the orbital referential), the SwissCube disposes of three magnetotorquers and an inertia wheel. The magnetotorquers can generate a magnetic field that interacts with the local Earth field to create a small control torque; howerver, this torque is constrained to the plane perpendicular to the local field, always leaving an uncontrollable direction.
The design of a suitable control law involves non-linear and non-autonomous system theory.

The appendix present some of the theoretical tools used.

2 Referentials

Before we derive the dynamics model of the satellite, we need to introduce the global geometry and introduce some notations. The SwissCube orbit will be almost circular and sun-synchronous (passing near the poles). It can be sketched as follows.



We will deal with three different reference frames. An inertial one, fixed to the Earth, an orbital one (ORF), fixed to the orbit with the positive x-direction pointing in the direction of displacement and a positive z-direction pointing toward the center of the Earth. The last one is the body-fixed referential (BRF). It coincides with the ORF when the satellite has the desired nominal orientation.
A vector will be denoted by $\vec{x}'$ when expressed in the BRF and by $\vec{x}$ when expressed in the ORF (sometimes also in the IRF, when this is clear from the context). When confusion is possible between IRF and ORF, the notation $\vec{x}_{IRF}$ is used to make the distinction.

Under the term dynamics, we understand the dynamics of the satellite orientation (i.e. attitude) and not the orbital dynamics on which we have no control. The orbital position and speed is considered a pure function of time in this work.

Also, we deal only with the control problem and the state variables are supposed to be known at each time.

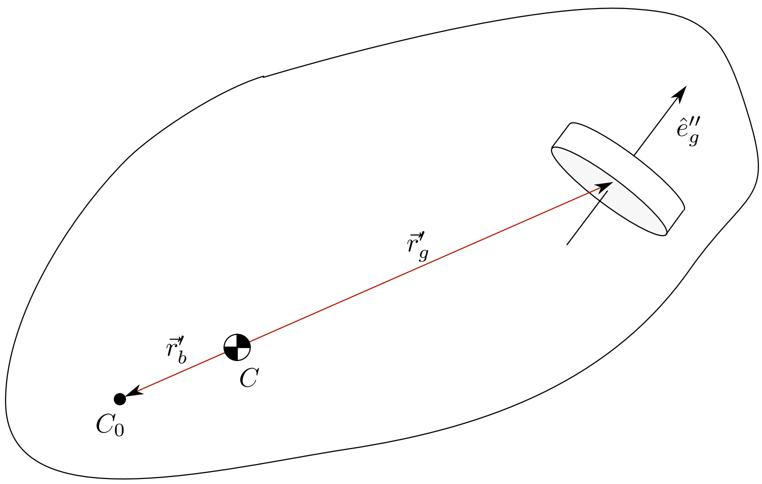
3 Dynamics Model

3.1 Inertial CubeSat Model

Here we establish the quaternion model of the satellite in an inertial reference frame. This is useful to get a first idea and will be used in the validation of the (more complex) non inertial model. This development is very similar to the one in [7].

3.1.1 Inertia Matrices and Reference Frames

Before we derive the dynamical model of the CubeSat, we need to describe how the inertia matrix of each element is defined. The following figure shows the center of gravity (CoG) of the whole satellite C, the CoG of the satellite without inertia wheel C_0, the vector $\vec{r}_b'$ linking them. $\vec{r}_g'$ is the vector linking C and the wheel's CoG. $\hat{e}_g''$ is the vector giving the wheel's rotation axis. Vectors expressed in the body (satellite) reference frame are noted with a $'$ and vectors in the wheel's reference frame are noted with $''$. Howerver, $\hat{e}_g''$ will simply be noted $\hat{e}_g$.



The following table gives the symbols used and their meaning

J_0	Body inertia without wheel	J_r	Inertia due to displacement of $C_0 \rightarrow C$
I_0	Wheel inertia	I_r	Inertia due to displacement $\vec{r}_g'$ of the wheels CoG
m_b	Body mass without wheel	m_g	Wheel mass
Φ	Transformation between wheel ref. and body ref. $\vec{x}' = \Phi\vec{x}''$		
$\hat{e}_g$	Wheel rotation axis in wheels ref.	$\vec{\omega}_g''$	$\vec{\omega}_g'' = \omega_g \hat{e}_g$ Wheel speed
$\vec{\omega}'$	Body rotational speed, relative to inertial ref.		

Note that because of the wheels symmetry around its axis of rotation, we can avoid to make the wheels referential be in rotation with the wheel. Therefore,

Φ is a constant matrix.
Using the parallel axes theorem [8] [1], we may write

$$\begin{aligned} J_r &= m_b[\vec{r'}_b^T \vec{r'}_b I_3 - \vec{r'}_b \vec{r'}_b^T] \\ I_r &= m_g[\vec{r'}_g^T \vec{r'}_g I_3 - \vec{r'}_g \vec{r'}_g^T]. \end{aligned}$$

3.1.2 Lagrangian

When dealing with potential free rotating motion, the Lagrangian L is nothing else but the rotational kinetic energy E. The kinetic energy is the sum of the kinetic energy of each rotating part

$$\begin{aligned} L = E &= \frac{1}{2}\vec{\omega}'^T J_0 \vec{\omega}' + \frac{1}{2}\vec{\omega}'^T J_r \vec{\omega}' + \frac{1}{2}\vec{\omega}'^T I_r \vec{\omega}' + \underbrace{\frac{1}{2}(\Phi^T \vec{\omega}' + \vec{\omega}''_g)^T I_0 (\Phi^T \vec{\omega}' + \vec{\omega}''_g)}_{\circledast} \\ \circledast &= \frac{1}{2}\vec{\omega}'^T \Phi I_0 \Phi^T \vec{\omega}' + \frac{1}{2}\vec{\omega}''^T_g I_0 \vec{\omega}' + \frac{1}{2}\vec{\omega}'^T \Phi I_0 \vec{\omega}''_g + \underbrace{\frac{1}{2}\vec{\omega}''^T_g I_0 \Phi^T \vec{\omega}'}_{\frac{1}{2}\vec{\omega}'^T \Phi I_0 \vec{\omega}''_g} \\ L &= \frac{1}{2}\vec{\omega}'^T \underbrace{(J_0 + J_r + I_r + \Phi I_0 \Phi^T)}_{J} \vec{\omega}' + \frac{1}{2}(2\Phi^T \vec{\omega}' + \vec{\omega}''_g)^T I_0 (\vec{\omega}''_g) \end{aligned}$$

That is

$$L = \frac{1}{2}\vec{\omega}'^T J \vec{\omega}' + \frac{1}{2}(2\Phi^T \vec{\omega}' + \vec{\omega}''_g)^T I_0 (\vec{\omega}''_g)$$

$$\boxed{L = \tfrac{1}{2}\vec{\omega}'^T J \vec{\omega}' + \tfrac{1}{2}(2\Phi^T \vec{\omega}' + \omega_g \hat{e}_g)^T I_0 (\omega_g \hat{e}_g)}. \tag{1}$$

J can be interpreted as the inertia matrix of the satellite *with* non-rotating (stopped) wheel.

3.1.3 Lagrangian Derivatives

Using the quaternion vector $\mathbf{q}$ and the wheel rotation velocity ω_g as coordinates, we need to compute $\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{c}}}$ and $\frac{\partial L}{\partial \mathbf{c}}$, with $\mathbf{c} = (\mathbf{q}^T \omega_g)^T$.
To derive L with respect to $\dot{\mathbf{q}}$ we first rewrite L remembering that $\vec{\omega}' = 2G\dot{\mathbf{q}}$ (57)

$$\begin{aligned} L &= 2(G\dot{\mathbf{q}})^T J (G\dot{\mathbf{q}}) + \frac{1}{2}(4\Phi^T G\dot{\mathbf{q}} + \omega_g \hat{e}_g)^T I_0 (\omega_g \hat{e}_g) \\ &= 2\dot{\mathbf{q}}^T (G^T J G)\dot{\mathbf{q}} + 2\omega_g \dot{\mathbf{q}}^T G^T \Phi I_0 \hat{e}_g + \frac{1}{2}\omega_g^2 \hat{e}_g^T I_0 \hat{e}_g. \end{aligned}$$

Thus

[1] http://en.wikipedia.org/wiki/Moment_of_inertia_tensor

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = 4G^T JG\dot{\mathbf{q}} + 2\omega_g G^T \Phi I_0 \hat{e}_g$$
$$= 2G^T J\vec{\omega}' + 2\omega_g G^T \Phi I_0 \hat{e}_g$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = 2\dot{G}^T J\vec{\omega}' + 2G^T J\dot{\vec{\omega}}' + 2\omega_g \dot{G}^T \Phi I_0 \hat{e}_g + 2\dot{\omega}_g G^T \Phi I_0 \hat{e}_g. \quad (2)$$

To derive L with respect to $\mathbf{q}$ we first rewrite L remembering that $\vec{\omega}' = -2\dot{G}\mathbf{q}$ [(57)]

$$L = 2(\dot{G}\mathbf{q})^T J(\dot{G}\mathbf{q}) + \frac{1}{2}(-4\Phi^T \dot{G}\mathbf{q} + \omega_g \hat{e}_g)^T I_0(\omega_g \hat{e}_g)$$
$$= 2\mathbf{q}^T(\dot{G}^T J\dot{G})\mathbf{q} - 2\omega_g \mathbf{q}^T \dot{G}^T \Phi I_0 \hat{e}_g + \frac{1}{2}\omega_g^2 \hat{e}_g^T I_0 \hat{e}_g.$$

Therefore

$$\frac{\partial L}{\partial \mathbf{q}} = 4\dot{G}^T J\dot{G}\mathbf{q} - 2\omega_g \dot{G}^T \Phi I_0 \hat{e}_g$$
$$\frac{\partial L}{\partial \mathbf{q}} = -2\dot{G}^T J\vec{\omega}' - 2\omega_g \dot{G}^T \Phi I_0 \hat{e}_g. \quad (3)$$

The angular position of the wheel ρ is also a coordinate, but because we are not interested in it and because $\frac{\partial L}{\partial \rho} = 0$, we don't need to investigate it further. However

$$\frac{\partial L}{\partial \dot{\rho}} = \frac{\partial L}{\partial \omega_g} = \vec{\omega}'^T \Phi I_o \hat{e}_g + \hat{e}_g^T I_0 \hat{e}_g \omega_g$$
$$= (\vec{\omega}'^T \Phi + \omega_g \hat{e}_g^T) I_0 \hat{e}_g$$

does not vanish and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} = \frac{d}{dt}\frac{\partial L}{\partial \omega_g} = (\dot{\vec{\omega}}'^T \Phi + \dot{\omega}_g \hat{e}_g^T) I_0 \hat{e}_g = \hat{e}_g^T I_0 (\Phi^T \dot{\vec{\omega}}' + \dot{\omega}_g \hat{e}_g). \quad (4)$$

3.1.4 Generalized Forces and Constraints

As in (68) and for the same reasons, the generalized force in the quaternion coordinates is $\mathbf{F_q} = 2G^T \vec{T}'$. On the other hand, the generalized force on the ρ coordinate is simply T_g, the momentum applied to the wheel.
Note that T_g must not be substracted from $\vec{T}'$ as what one could think of as a reaction torque; because this reaction torque does *not* perform any work when $\mathbf{q}$ is varying and ρ freezed. This is the same as for all the structural forces within the body.
The constraint is also $C = \mathbf{q}^T\mathbf{q} = 1$ here, and $\frac{\partial C}{\partial \mathbf{q}} = 2\mathbf{q}$, written as column vector.

3.1.5 Dynamics

From the Lagrange formulation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{c}}} - \frac{\partial L}{\partial \mathbf{c}} = \mathbf{F_c} - \lambda \frac{\partial C}{\partial \mathbf{c}},$$

we get for the quaternion part of $\mathbf{c}$ from (2) and (3)

$$\begin{array}{rlll} & 2\dot{G}^T J\vec{\omega}' + 2G^T J\dot{\vec{\omega}}' & + & 2\omega_g \dot{G}^T \Phi I_0 \hat{e}_g + 2\dot{\omega}_g G^T \Phi I_0 \hat{e}_g \\ + & 2\dot{G}^T J\vec{\omega}' & + & 2\omega_g \dot{G}^T \Phi I_0 \hat{e}_g \qquad = 2G^T \vec{T}' + 2\lambda \mathbf{q}, \end{array}$$

hence

$$4\dot{G}^T J\vec{\omega}' + 2G^T J\dot{\vec{\omega}}' + 4\omega_g \dot{G}^T \Phi I_0 \hat{e}_g + 2\dot{\omega}_g G^T \Phi I_0 \hat{e}_g = 2G^T \vec{T}' + 2\lambda \mathbf{q}.$$

Dividing by 2 and left multiplying by G, we get

$$\underbrace{2G\dot{G}^T}_{\Omega'} J\vec{\omega}' + \underbrace{GG^T}_{Id} J\dot{\vec{\omega}}' + \omega_g \underbrace{2G\dot{G}^T}_{\Omega'} \Phi I_0 \hat{e}_g + \dot{\omega}_g \underbrace{GG^T}_{Id} \Phi I_0 \hat{e}_g = \underbrace{GG^T}_{Id} \vec{T}' + \lambda \underbrace{G\mathbf{q}}_{\vec{0}}$$

$$\Omega'(J\vec{\omega}' + \omega_g \Phi I_0 \hat{e}_g) + J\dot{\vec{\omega}}' + \dot{\omega}_g \Phi I_0 \hat{e}_g = \vec{T}'.$$

Remembering from (61) that $\Omega'\vec{v} = \vec{\omega}' \times \vec{v} \;\; \forall \vec{v}$,

$$\vec{\omega}' \times (J\vec{\omega}' + \omega_g \Phi I_0 \hat{e}_g) + J\dot{\vec{\omega}}' + \dot{\omega}_g \Phi I_0 \hat{e}_g = \vec{T}'$$

That is

$$J\dot{\vec{\omega}}' + \dot{\omega}_g \Phi I_0 \hat{e}_g = \vec{T}' - \vec{\omega}' \times (J\vec{\omega}' + \omega_g \Phi I_0 \hat{e}_g). \tag{5}$$

From Lagrange and (4), we have

$$\hat{e}_g^T I_0 \Phi^T \dot{\vec{\omega}}' + \dot{\omega}_g \hat{e}_g^T I_0 \hat{e}_g = T_g. \tag{6}$$

Together with (5) we can now write the full satellite dynamics in an inertial reference frame

$$\underbrace{\begin{pmatrix} J & \Phi I_0 \hat{e}_g \\ \hat{e}_g^T I_0 \Phi^T & \hat{e}_g^T I_0 \hat{e}_g \end{pmatrix}}_{H^{-1}} \begin{pmatrix} \dot{\vec{\omega}}' \\ \dot{\omega}_g \end{pmatrix} = \begin{pmatrix} \vec{T}' - \vec{\omega}' \times (J\vec{\omega}' + \omega_g \Phi I_0 \hat{e}_g) \\ T_g \end{pmatrix}$$

$$\boxed{\begin{array}{c} \begin{pmatrix} \dot{\vec{\omega}}' \\ \dot{\omega}_g \end{pmatrix} = H \begin{pmatrix} \vec{T}' - \vec{\omega}' \times (J\vec{\omega}' + \omega_g \Phi I_0 \hat{e}_g) \\ T_g \end{pmatrix} \\ \dot{\mathbf{q}} = \frac{1}{2} G^T \vec{\omega}'. \end{array}} \tag{7}$$

And because H is constant, it can be pre-computed for faster simulation.

3.2 Non-Inertial CubeSat Model

Here we establish the quaternion model of the satellite in a non-inertial reference frame, namely the orbital reference frame (ORF). $\vec{\omega}_o$ represents the rotational velocity of the ORF expressed in the ORF. $\vec{\omega}'$ is now the satellite speed relative to the ORF expressed in the body referential.

3.2.1 Lagrangian

As in the inertial model, the Lagrangian is simply the rotational kinetic energy. (1) still applies, but we have to replace the rotation speed in it by $\vec{\omega}'_I$, to make clear that it must be a velocity that is measured relative to an inertial frame. So (1) becomes

$$L = \frac{1}{2}\vec{\omega}_I'^T J\vec{\omega}'_I + \frac{1}{2}(2\Phi^T\vec{\omega}'_I + \vec{\omega}''_g)^T I_0(\vec{\omega}''_g). \tag{8}$$

Now we express $\vec{\omega}'_I$ in function of $\vec{\omega}'$. The quaternion q describes the body attitude relative to the ORF. As shown in C by speed composition and with $R^T = GE^T$ we can write

$$\vec{\omega}'_I = \vec{\omega}' + R^T\vec{\omega}_o = 2G\dot{\mathbf{q}} + R^T\vec{\omega}_o = -2\dot{G}\mathbf{q} + R^T\vec{\omega}_o.$$

Substituting by the second equality into (8), we get

$$L = \frac{1}{2}(2G\dot{\mathbf{q}} + R^T\vec{\omega}_o)^T J(2G\dot{\mathbf{q}} + R^T\vec{\omega}_o) + \frac{1}{2}(2\Phi^T(2G\dot{\mathbf{q}} + R^T\vec{\omega}_o) + \vec{\omega}''_g)^T I_0(\vec{\omega}''_g).$$

Expanding

$$\begin{aligned}
L &= 2\dot{\mathbf{q}}^T(G^TJG)\dot{\mathbf{q}} + \frac{1}{2}\vec{\omega}_o^T(RJR^T)\vec{\omega}_o + \dot{\mathbf{q}}^T(G^TJR^T)\vec{\omega}_o + \vec{\omega}_o^T(RJG)\dot{\mathbf{q}} \\
&+ \frac{1}{2}(2\Phi^T(2G\dot{\mathbf{q}} + R^T\vec{\omega}_o) + \vec{\omega}''_g)^T I_0(\vec{\omega}''_g) \\
&= 2\dot{\mathbf{q}}^T(G^TJG)\dot{\mathbf{q}} + \frac{1}{2}\vec{\omega}_o^T(RJR^T)\vec{\omega}_o + 2\dot{\mathbf{q}}^T(G^TJR^T)\vec{\omega}_o \\
&+ (2\Phi^TG\dot{\mathbf{q}})^T I_0\vec{\omega}''_g + (\Phi^TR^T\vec{\omega}_o)^T I_0\vec{\omega}''_g + \frac{1}{2}\vec{\omega}_g''^T I_0\vec{\omega}''_g
\end{aligned}$$

$$\begin{aligned}
L &= 2\dot{\mathbf{q}}^T(G^TJG)\dot{\mathbf{q}} + \frac{1}{2}\vec{\omega}_o^T(RJR^T)\vec{\omega}_o + 2\dot{\mathbf{q}}^T(G^TJR^T)\vec{\omega}_o \\
&+ 2\dot{\mathbf{q}}^T(G^T\Phi I_0)\vec{\omega}''_g + \vec{\omega}_o^T(R\Phi I_0)\vec{\omega}''_g + \frac{1}{2}\vec{\omega}_g''^T I_0\vec{\omega}''_g
\end{aligned} \tag{9}$$

Replacing $2G\dot{\mathbf{q}}$ by $-2\dot{G}\mathbf{q}$ gives an other version of the Lagrangian

$$\begin{aligned}
L &= 2\mathbf{q}^T(\dot{G}^TJ\dot{G})\mathbf{q} + \frac{1}{2}\vec{\omega}_o^T(RJR^T)\vec{\omega}_o - 2\mathbf{q}^T(\dot{G}^TJR^T)\vec{\omega}_o \\
&- 2\mathbf{q}^T(\dot{G}^T\Phi I_0)\vec{\omega}''_g + \vec{\omega}_o^T(R\Phi I_0)\vec{\omega}''_g + \frac{1}{2}\vec{\omega}_g''^T I_0\vec{\omega}''_g
\end{aligned} \tag{10}$$

3.2.2 Derivative by q

Using (70) through (77) in Appendix B.1 and from (10), we can write

$$
\begin{aligned}
\frac{\partial L}{\partial \mathbf{q}} &= 4(\dot{G}^T J \dot{G})\mathbf{q} + 2\Delta[\vec{\omega}_o, JR^T\vec{\omega}_o]\mathbf{q} - 2(\dot{G}^T JR^T)\vec{\omega}_o \\
&- 4\Delta[\vec{\omega}_o, J\dot{G}\mathbf{q}]\mathbf{q} - 2(\dot{G}^T\Phi I_0)\vec{\omega}''_g + 2\Delta[\vec{\omega}_o, \Phi I_o\vec{\omega}''_g]\mathbf{q} \\
&= 4(\dot{G}^T J \dot{G})\mathbf{q} - 2(\dot{G}^T JR^T)\vec{\omega}_o - 2(\dot{G}^T\Phi I_0)\vec{\omega}''_g \\
&+ 2\Delta[\vec{\omega}_o, JR^T\vec{\omega}_o - 2J\dot{G}\mathbf{q} + \Phi I_o\vec{\omega}''_g]\mathbf{q} \\
&= -2(\dot{G}^T J)\vec{\omega}' - 2(\dot{G}^T JR^T)\vec{\omega}_o - 2(\dot{G}^T\Phi I_0)\vec{\omega}''_g \\
&+ 2\Delta[\vec{\omega}_o, JR^T\vec{\omega}_o - 2J\dot{G}\mathbf{q} + \Phi I_o\vec{\omega}''_g]\mathbf{q} \\
&= -2\dot{G}^T(J\vec{\omega}' + JR^T\vec{\omega}_o + \Phi I_0\vec{\omega}''_g) \\
&+ 2\Delta[\vec{\omega}_o, JR^T\vec{\omega}_o - 2J\dot{G}\mathbf{q} + \Phi I_o\vec{\omega}''_g]\mathbf{q}. \qquad (11)
\end{aligned}
$$

3.2.3 Derivative by $\dot{\mathbf{q}}$ and t

This time we start from the Lagrangian in the form given in (9)

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\mathbf{q}}} &= 4(G^T JG)\dot{\mathbf{q}} + 2(G^T JR^T)\vec{\omega}_o + 2(G^T\Phi I_0)\vec{\omega}''_g \\
&= 2(G^T J)\vec{\omega}' + 2(G^T JR^T)\vec{\omega}_o + 2(G^T\Phi I_0)\vec{\omega}''_g \\
&= 2G^T(J\vec{\omega}' + JR^T\vec{\omega}_o + \Phi I_0\vec{\omega}''_g) \qquad (12)
\end{aligned}
$$

Now, when taking the time derivative of (12), we stumble on the time derivative of the rotation matrix R. Using $\dot{R}^T = -\Omega' R^T$, from Appendix B.2, we can write

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} &= 2\dot{G}^T(J\vec{\omega}' + JR^T\vec{\omega}_o + \Phi I_0\vec{\omega}''_g) \\
&+ 2G^T(J\dot{\vec{\omega}}' + J\dot{R}^T\vec{\omega}_o + JR^T\dot{\vec{\omega}}_o + \Phi I_0\dot{\vec{\omega}}''_g) \\
&= 2\dot{G}^T(J\vec{\omega}' + JR^T\vec{\omega}_o + \Phi I_0\vec{\omega}''_g) \\
&+ 2G^T(J\dot{\vec{\omega}}' - J\Omega' R^T\vec{\omega}_o + JR^T\dot{\vec{\omega}}_o + \dot{\omega}_g\Phi I_0\hat{e}_g) \qquad (13)
\end{aligned}
$$

3.2.4 Dynamics in $\vec{\omega}'$

Putting everything in the Lagragian formulation for the sub-dynamics in the coordinate quaternion

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{F}_{\mathbf{q}} - \lambda\frac{\partial C}{\partial \mathbf{q}}
$$

we get

$$
\begin{aligned}
&2\dot{G}^T(J\vec{\omega}' + JR^T\vec{\omega}_o + \Phi I_0\vec{\omega}_g'')\\
&+2G^T(J\dot{\vec{\omega}}' - J\Omega' R^T\vec{\omega}_o + JR^T\dot{\vec{\omega}}_o + \dot{\omega}_g\Phi I_0\hat{e}_g)\\
&+2\dot{G}^T(J\vec{\omega}' + JR^T\vec{\omega}_o + \Phi I_0\vec{\omega}_g'')\\
&-2\Delta[\vec{\omega}_o, JR^T\vec{\omega}_o - 2J\dot{G}\mathbf{q} + \Phi I_o\vec{\omega}_g'']\mathbf{q}\\
&=2G^T\vec{T}' + 2\lambda\mathbf{q}.
\end{aligned}
$$

With $\vec{u} := J\vec{\omega}' + JR^T\vec{\omega}_o + \Phi I_0\vec{\omega}_g''$ we have

$$
\begin{aligned}
&4\dot{G}^T\vec{u} - 2\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q}\\
&+2G^T(J\dot{\vec{\omega}}' - J\Omega' R^T\vec{\omega}_o + JR^T\dot{\vec{\omega}}_o + \dot{\omega}_g\Phi I_0\hat{e}_g)\\
&=2G^T\vec{T}' + 2\lambda\mathbf{q}.
\end{aligned}
$$

Dividing by 2 and left multiplying by G

$$
\begin{aligned}
&2G\dot{G}^T\vec{u} - G\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q}\\
&+GG^T(J\dot{\vec{\omega}}' - J\Omega' R^T\vec{\omega}_o + JR^T\dot{\vec{\omega}}_o + \dot{\omega}_g\Phi I_0\hat{e}_g)\\
&=GG^T\vec{T}' + \lambda G\mathbf{q}.
\end{aligned}
$$

Using $2G\dot{G}^T\vec{u} = \Omega'\vec{u} = \vec{\omega}' \times \vec{u}$, $GG^T = Id$ and $G\mathbf{q} = \vec{0}$ from appendix A

$$
\begin{aligned}
&\vec{\omega}' \times \vec{u} - G\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q}\\
&+J\dot{\vec{\omega}}' - J\Omega' R^T\vec{\omega}_o + JR^T\dot{\vec{\omega}}_o + \dot{\omega}_g\Phi I_0\hat{e}_g = \vec{T}'
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
J\dot{\vec{\omega}}' + \dot{\omega}_g\Phi I_0\hat{e}_g = &\vec{T}' - \vec{\omega}' \times \vec{u} + G\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q}\\
&+ J\Omega' R^T\vec{\omega}_o - JR^T\dot{\vec{\omega}}_o.
\end{aligned}
$$

With $\vec{s} := -\Omega' R^T\vec{\omega}_o + R^T\dot{\vec{\omega}}_o = (R^T\vec{\omega}_o) \times \vec{\omega}' + R^T\dot{\vec{\omega}}_o$ we finally have

$$
J\dot{\vec{\omega}}' + \dot{\omega}_g\Phi I_0\hat{e}_g = \vec{T}' - \vec{\omega}' \times \vec{u} + G\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q} - J\vec{s}. \tag{14}
$$

3.2.5 Wheel Dynamics

We now do the same, but for the wheel's angular position coordinate ρ_g. As in the inertial model, $\frac{\partial L}{\partial \rho_g} = 0$. We first compute $\frac{\partial L}{\partial \dot{\rho}_g} = \frac{\partial L}{\partial \omega_g}$ starting from (9)

$$
\begin{aligned}
\frac{\partial L}{\partial \omega_g} &= 2\dot{\mathbf{q}}^T(G^T\Phi I_0)\hat{e}_g + \vec{\omega}_o^T(R\Phi I_0)\hat{e}_g + \hat{e}_g I_0\hat{e}_g\omega_g\\
&= \vec{\omega}'^T GG^T\Phi I_0\hat{e}_g + \vec{\omega}_o^T(R\Phi I_0)\hat{e}_g + \hat{e}_g I_0\hat{e}_g\omega_g\\
&= \vec{\omega}'^T(\Phi I_0)\hat{e}_g + \vec{\omega}_o^T(R\Phi I_0)\hat{e}_g + \hat{e}_g I_0\hat{e}_g\omega_g\\
&= \hat{e}_g^T I_0\Phi^T\vec{\omega}' + \hat{e}_g^T I_0\Phi^T R^T\vec{\omega}_o + \hat{e}_g I_0\hat{e}_g\omega_g\\
&= \hat{e}_g^T I_0(\Phi^T\vec{\omega}' + \Phi^T R^T\vec{\omega}_o + \omega_g\hat{e}_g).
\end{aligned}
$$

Taking the time derivative one finds

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \omega_g} &= \hat{e}_g^T I_0(\Phi^T \dot{\vec{\omega}}' + \Phi^T(-\Omega' R^T \vec{\omega}_o + R^T \dot{\vec{\omega}}_o) + \dot{\omega}_g \hat{e}_g)\\ &= \hat{e}_g^T I_0(\Phi^T \dot{\vec{\omega}}' + \Phi^T \vec{s} + \dot{\omega}_g \hat{e}_g).\end{aligned}$$

and therefore

$$\frac{d}{dt}\frac{\partial L}{\partial \omega_g} = T_g$$

giving

$$\hat{e}_g^T I_0(\Phi^T \dot{\vec{\omega}}' + \Phi^T \vec{s} + \dot{\omega}_g \hat{e}_g) = T_g$$

$$\hat{e}_g^T I_0 \Phi^T \dot{\vec{\omega}}' + \hat{e}_g^T I_0 \hat{e}_g \dot{\omega}_g = T_g - \hat{e}_g^T I_0 \Phi^T \vec{s}. \tag{15}$$

3.2.6 Complete Satellite Dynamics in Non Inertial Frame

Putting together both dynamics developed before, we can write the complete satellite dynamics in non inertial frame (ORF)

$$\underbrace{\begin{pmatrix} J & \Phi I_0 \hat{e}_g \\ \hat{e}_g^T I_0 \Phi^T & \hat{e}_g^T I_0 \hat{e}_g \end{pmatrix}}_{H^{-1}} \begin{pmatrix} \dot{\vec{\omega}}' \\ \dot{\omega}_g \end{pmatrix} = \begin{pmatrix} \vec{T}' - \vec{\omega}' \times \vec{u} + G\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q} - J\vec{s} \\ T_g - \hat{e}_g^T I_0 \Phi^T \vec{s} \end{pmatrix}$$

$$\boxed{\begin{gathered} \begin{pmatrix} \dot{\vec{\omega}}' \\ \dot{\omega}_g \end{pmatrix} = H \begin{pmatrix} \vec{T}' - \vec{\omega}' \times \vec{u} + G\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q} - J\vec{s} \\ T_g - \hat{e}_g^T I_0 \Phi^T \vec{s} \end{pmatrix} \\[2ex] \dot{\mathbf{q}} = \tfrac{1}{2} G^T \vec{\omega}'. \\[2ex] \vec{u} := J\vec{\omega}' + JR^T \vec{\omega}_o + \Phi I_0 \vec{\omega}_g'' \\ \vec{s} := (R^T \vec{\omega}_o) \times \vec{\omega}' + R^T \dot{\vec{\omega}}_o \end{gathered}} \tag{16}$$

with

$$\Delta[\vec{v}, \vec{w}] = \begin{pmatrix} \vec{w} \cdot \vec{v} & (\vec{w} \times \vec{v})^T \\ \vec{w} \times \vec{v} & \vec{w}\vec{v}^T + \vec{v}\vec{w}^T - \vec{w} \cdot \vec{v}\ I_3 \end{pmatrix}.$$

3.2.7 Satellite Dynamics in Non Inertial Frame Without Wheel

One last simplified version of the model is the following. The inertia wheel is taken out and the ORF speed is considered constant ($\dot{\vec{\omega}}_o = 0$). We do not repeat all the calculation steps, as they are similar to the previous ones.

$$\boxed{\begin{array}{c}\dot{\vec{\omega}}' = J^{-1}\vec{T}' - J^{-1}(\vec{\omega}' \times \vec{u}) + J^{-1}G\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q} - \vec{s} \\ \\ \dot{\mathbf{q}} = \frac{1}{2}G^T\vec{\omega}'. \\ \\ \vec{u} := J(\vec{\omega}' + R^T\vec{\omega}_o) \\ \vec{s} := (R^T\vec{\omega}_o) \times \vec{\omega}' \end{array}} \tag{17}$$

3.3 Euler Angles Non Inertial Model

For model validation, a model in non-inertial (ORF) frame with wheel has been derived by means of computer symbolic calculation. The resulting model is ugly, but is also supposed free of calculation errors. Validation is then performed by comparing simulation results with the quaternionic model. We give here a short description of the Euler model.
The rotation matrix R is calculated from $R = R_\phi R_\theta R_\psi$. R_ϕ, R_θ and R_ψ are defined as in Appendix D. Each rotation axis is then given by (in body coordinates)

$$\hat{e}_\phi = R_\psi R_\theta R_\psi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \hat{e}_\theta = R_\psi R_\theta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \hat{e}_\psi = R_\psi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The rotational velocity is therefore

$$\vec{\omega}' = \dot{\phi}\hat{e}_\phi + \dot{\theta}\hat{e}_\theta + \dot{\psi}\hat{e}_\psi.$$

The Lagrangian is the same as in (8) and the substitution $\vec{\omega}'_I = \vec{\omega}' + R^T\vec{\omega}_o$ is made. In order to the make the Lagrangian derivatives computable by computer, the expression

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{c}}} - \frac{\partial L}{\partial \mathbf{c}} = \mathbf{F_c}$$

with $\mathbf{c} = (\phi \;\, \theta \;\, \psi \;\, \rho_g)^T$, $\dot{\mathbf{c}} = (\dot{\phi} \;\, \dot{\theta} \;\, \dot{\psi} \;\, \omega_g)^T$ and $\mathbf{F_c} = (\vec{T}'^T \quad T_g)^T$ has to be broken down in the following way:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{c}}} = \frac{\partial^2 L}{\partial \dot{\mathbf{c}}^2}\ddot{\mathbf{c}} + \frac{\partial^2 L}{\partial \dot{\mathbf{c}}\partial \mathbf{c}}\dot{\mathbf{c}}$$

that is

$$\frac{\partial^2 L}{\partial \dot{\mathbf{c}}^2}\ddot{\mathbf{c}} + \frac{\partial^2 L}{\partial \dot{\mathbf{c}}\partial \mathbf{c}}\dot{\mathbf{c}} - \frac{\partial L}{\partial \mathbf{c}} = \mathbf{F_c}.$$

Rearranging the expression, we get the usual form for the dynamics

$$\frac{\partial^2 L}{\partial \dot{\mathbf{c}}^2}\ddot{c} = \mathbf{F_c} + \frac{\partial L}{\partial \mathbf{c}} - \frac{\partial^2 L}{\partial \dot{c}\partial c}\dot{c}.$$

The trivial substitution $\mathbf{c}_1 = \mathbf{c}$, $\mathbf{c}_2 = \dot{\mathbf{c}}$ and $\dot{\mathbf{c}}_1 = \mathbf{c}_2$ transforms the system to a first order one.
Note that the Hessian matrix $\frac{\partial^2 L}{\partial \dot{\mathbf{c}}^2}$ is not constant in the Euler model and has to be evaluated and inverted at each integration step. It is also this matrix that is responsible for the singularities of the model.

3.4 Validation

The three models of the satellite (inertial quaternion based, non-inertial quaternion based and non-inertial Euler angles based) have been simulated two by two in order to compare the results obtained. The same initial conditions are used using the relation developed in Appendix D to translate Euler angles into quaternion. To compare the inertial versus the non inertial quaternion models, relation (84) for speed composition is used.
Figures 1 to 3 show some simulation results. All simulations have been tested with various torques applied on the satellite and wheel and with various vanishing and non-vanishing orbital referential speeds and accelerations ($\vec{\omega}_o$ and $\dot{\vec{\omega}}_o$).

3.5 Some Notes About Simulation

All simulations performed here where done using the `ode45` solver from MATLAB. This solver is a Runge-Kutta type of integrator and is well adapted for the treatment of non-stiff ordinary differential equations (ODE). However, in all quaternionic versions of the satellite model, the additional norm constraint on the quaternion-parameters vector $\|\mathbf{q}\| = 1$ shows up. Hence, we are in presence of a differential-algebraic equation (DAE) system.
For the sole purpose of model and control algorithm validation, this isn't an issue, as we perform only finite-time simulations and we can go on using solvers of the type of `ode45`. But in the perspective of long time simulations (as may occur in the design and operation of an observer), one must be careful. Indeed, if the quaternion initial conditions satisfy the norm constraint $\|\mathbf{q}(t = 0)\| = 1$, then

$$\dot{\mathbf{q}} = \frac{1}{2} G^T \vec{\omega}'$$

theoretically guaranties that $\|\mathbf{q}(t)\| = 1$ is satisfied for all $t > 0$. In simulation however, numerical errors may lead the norm of $\mathbf{q}$ to derive significantly for large t. The problem of numerically solving DAEs is a field in itself and is extensievely covered in the literature. See for example [15].

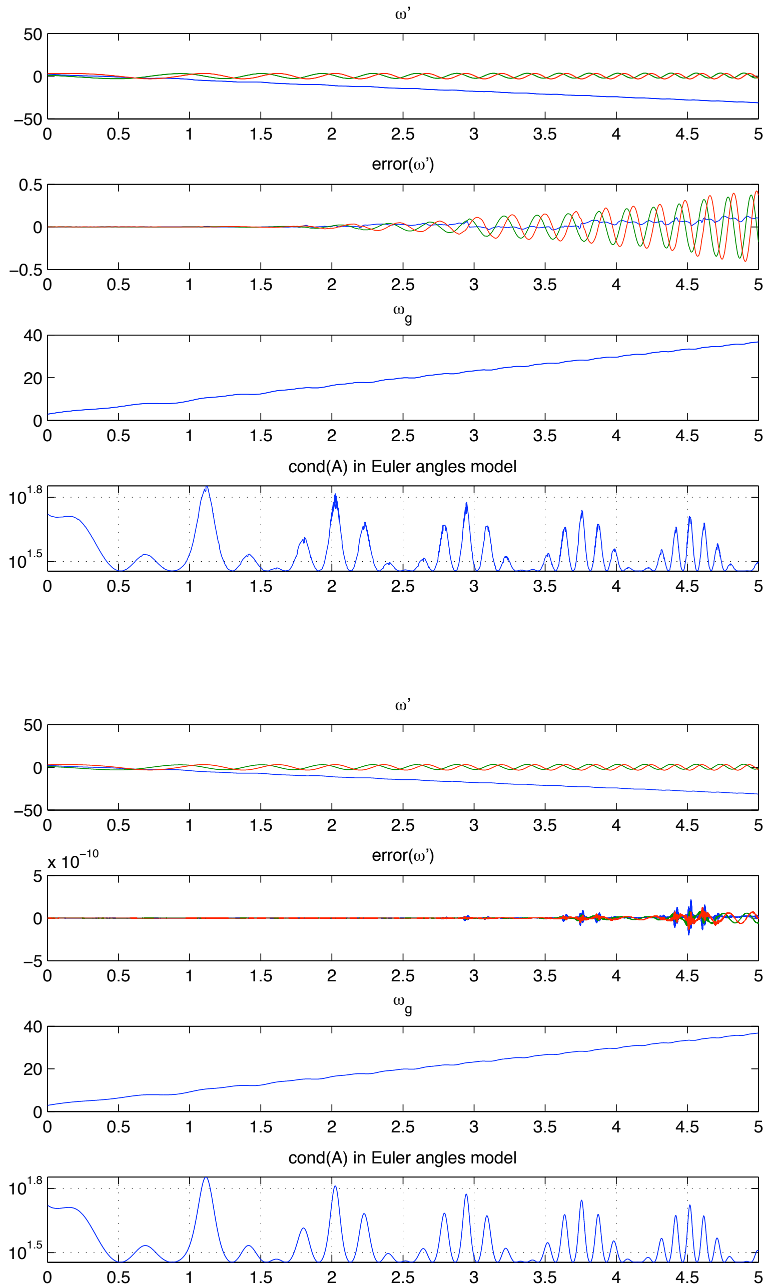


Figure 1: Comparison between Euler and quaternion models with increasing numerical precision (1000 steps upper graphs and 40000 bottom graphs). Observe the progressively increasing difference (error) as time goes. Last plot shows reasonable Hessian condition over all simulation time.

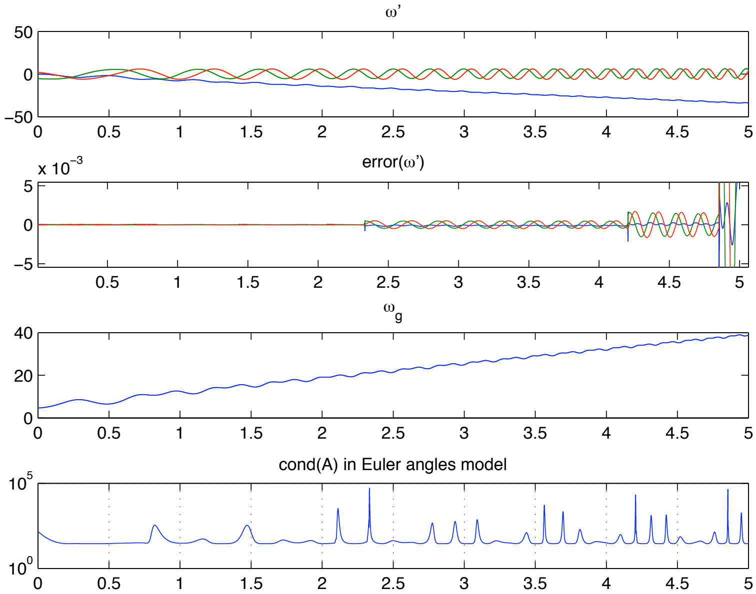


Figure 2: Comparison between Euler and quaternion models with high numerical resolution (40000 steps). This time, the initial conditions are such that the Hessian matrix condition reaches very high values many times. Observe how the error (difference) increases stepwise with each major condition peak. This shows the limitations of the Euler angles model.

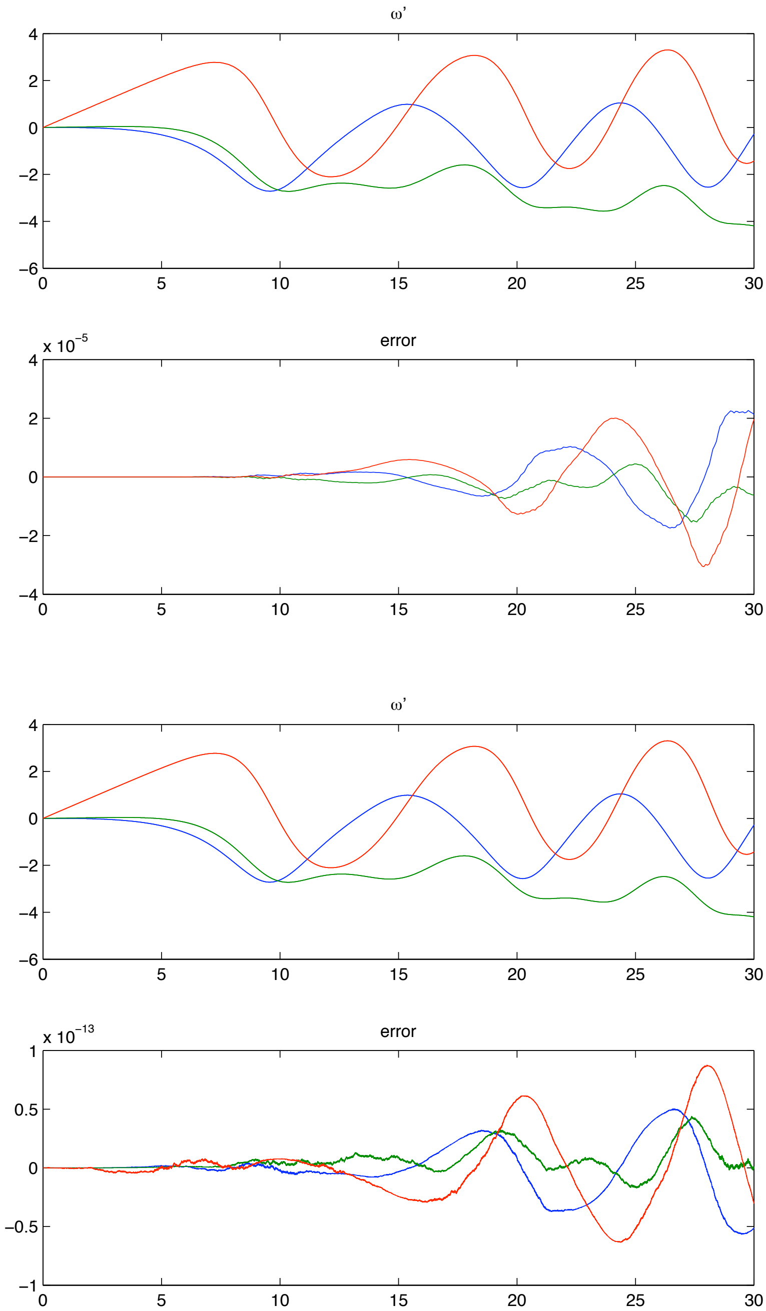


Figure 3: Comparison between inertial quaternion and non-inertial quaternion models with increasing numerical precision (417 steps upper graphs and 40000 bottom graphs). Note that the difference (error) increases regularly with time and decreases with time resolution; this tends to show that both models are behaving the same within numerical precision.

4 Controllability with Spinning Wheel

Due to its design, the inertia wheel has to be operated with a non zero average rotational speed. In this section, we study the effect of the speed of the wheel on the controllability matrix of the linearized system (around $\vec{\omega}' = \vec{0}$, $\mathbf{q} = (1\ 0\ 0\ 0)^T$ and $\omega_g = \bar{\omega}_g$).

Consider the system (16) and rewrite it as

$$\dot{x} = f(x, y)$$

with $x = (\vec{\omega}'^T\ \omega_g\ \mathbf{q}^T)^T$ the state variable and $y = (\vec{T}^T\ T_g)^T$ the control input.

Linearizing the system around the origin leads to the system matrices

$$A(\bar{\omega}_g) = A = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} = \mathbf{x}_0, y = 0) \qquad B(\bar{\omega}_g) = B = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{x} = \mathbf{x}_0, y = 0)$$

with $x_0 = (\vec{0}^T\ \bar{\omega}_g\ \mathbf{0}^T)^T$.

The matrices A and B are then evaluated for some values of $\bar{\omega}_g$. For each matrix pair obtained, the controllability matrix

$$C = [B\ \ AB\ \ ...\ \ A^7B]$$

is constructed. The condition of the problem is finally computed with the ratio of the singular values 1 and 7, $\frac{\sigma_1}{\sigma_7}$. Note that $\sigma_8 = 0$, because it correspond to the uncontrollability of the norm of $\mathbf{q}$ (i.e. the algebraic constraint on the coordinate quaternion).

The calculation are done in MATLAB and the Symbolic toolbox in the script `wheel_controllability.m`

4.1 Numerical Results

Using

$$J = 10^{-3}\begin{pmatrix} 1.845 & 0.08034 & -0.01125 \\ 0.08034 & 2.566 & -0.07533 \\ -0.01125 & -0.07533 & 2.309 \end{pmatrix} kg \cdot m$$

$$I_0 = 10^{-6}\begin{pmatrix} 10.66 & 0.0 & 0.0 \\ 0.0 & 10.66 & 0.0 \\ 0.0 & 0.0 & 21.22 \end{pmatrix} kg \cdot m$$

and with the wheel aligned on the body *y-axis* we obtain the controllability presented in Figure 4.

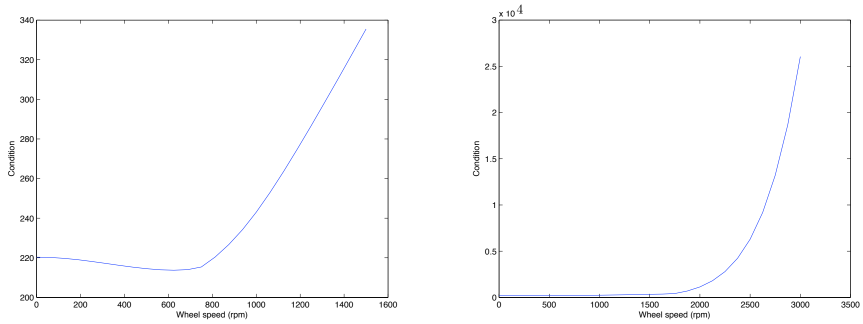


Figure 4: $\frac{\sigma_1(C)}{\sigma_7(C)}$ in function of mean inertia wheel speed $\bar{\omega}_g$

4.2 Implications

One can observe, that the condition of C is almost unchanged for wheel speeds up to about 1000rpm and then explodes for higher speeds. This means that controllability is lost for high rotating speeds. At the same time, around the origin (during nominal control), the y-direction is always controllable by the magnetotorquers, making the wheel useless anyway. Putting the wheel on an other axis is also problematic, as stabilization in the ORF implies a constant rotation speed along $\hat{e}_y$ (relative to an inertial frame).

For those reasons, it appears that a unique inertia wheel along the *y-axis* is of little benefit (if any). It must also be pointed out that its contribution to the overall dynamics was relatively easy to model (system 16) but that the resulting theoretical non-linear control problem becomes very difficult.

5 Control

5.1 Introduction

As seen before, the use of the inertial wheel for the attitude control is of limited interest. Therefore, we will now concentrate on what is feasible using the magnetotorquers only. Because we use no wheel (it can be onboard but must be stopped), the dynamics reduces to the one of a rigid body. We will first start with the dynamics expressed within an inertial reference frame to show the ideas and principles used.

5.2 Control Law on Inertial System

5.2.1 Dissipative Term

Consider the satellite dynamics expressed in an inertial frame with the control torque $\vec{T}' = -k_v\vec{\omega}'$ applied on it (see (69))

$$\boxed{\begin{array}{lcl} \dot{\vec{\omega}}' & = & -J^{-1}(k_v\vec{\omega}') - J^{-1}(\vec{\omega}' \times J\vec{\omega}') \\ \dot{\mathbf{q}} & = & \frac{1}{2}G^T\vec{\omega}'. \end{array}} \tag{18}$$

And consider the (energy) function

$$V = E_c = \frac{1}{2}\vec{\omega}'^T J\vec{\omega}' \geq 0. \tag{19}$$

The time derivative of V is then given by

$$\begin{aligned} \dot{V} &= \vec{\omega}'^T J\dot{\vec{\omega}}' \\ &= \vec{\omega}'^T J(-J^{-1}(k_v\vec{\omega}') - J^{-1}(\vec{\omega}' \times J\vec{\omega}')) \\ &= -k_v\vec{\omega}'^T\vec{\omega}' \leq 0. \end{aligned}$$

The set $\Omega = \{(\vec{\omega}', \mathbf{q}) : \|\vec{\omega}'\| \leq \omega_0, \|\mathbf{q}\| = 1\}$ is compact. Ω is also positively invariant because of the semi-negative definitness of $\dot{V} = \dot{V}(\vec{\omega}')$ with $V(\vec{\omega}')$ positive definite in $\vec{\omega}'$ and because of the norm constraint on $\mathbf{q}$.
For any $\vec{\omega}'(t = 0) \in \Omega$, the semi-definiteness of V and $\dot{V}$ allows us to invoke LaSalle's invariance principle. Hence we can conclude that the trajectories of (18) will tend to the largest invariant set, that is

$$(\vec{\omega}', \mathbf{q}) \to \{(\vec{\omega}', \mathbf{q}) : \vec{\omega}' = \vec{0}\} \qquad \text{as} \qquad t \to \infty.$$

This simply means, that with this control law, the satellite would eventually stop rotating, but would stop in any position.

5.2.2 Proportional Term

Remember the Lagrangian $L = E_c = \frac{1}{2}\vec{\omega}'^T J\vec{\omega}'$ used in Appendix A.6 to derive the dynamics (18). It is thus clear that (18) is a potential-free dynamics. We now propose to modify this Lagrangian by adding a potential term to it and to observe how it modifies the dynamics and the invariant set.
Take

$$L_2 = L - U(\mathbf{q}), \tag{20}$$

reapply the Lagrangian formalism to L_2 and get

$$\frac{d}{dt}\frac{\partial L_2}{\partial \dot{\mathbf{q}}} - \frac{\partial L_2}{\partial \mathbf{q}} = \mathbf{F}_q + \lambda \frac{\partial C}{\partial \mathbf{q}} \qquad \Leftrightarrow \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} + \nabla^T U = \mathbf{F}_q + \lambda \frac{\partial C}{\partial \mathbf{q}}.$$

From there, we go through the writing of the dynamics, as in Appendix A.6.3

$$4\dot{G}^T J\vec{\omega}' + 2G^T J\dot{\vec{\omega}}' + \nabla^T U = 2G^T \vec{T}' + \lambda \mathbf{q}.$$

Left-multiplying by G

$$\underbrace{4G\dot{G}^T}_{2\Omega'} J\vec{\omega}' + 2\underbrace{GG^T}_{Id} J\dot{\vec{\omega}}' + G\nabla^T U = 2\underbrace{GG^T}_{Id} \vec{T}' + \lambda \underbrace{G\mathbf{q}}_{\vec{0}}$$

$$2\dot{G}^T GJ\vec{\omega}' + GG^T J\dot{\vec{\omega}}' + \frac{1}{2}G\nabla^T U = GG^T \vec{T}'$$

$$\vec{\omega}' \times J\vec{\omega}' + J\dot{\vec{\omega}}' + \frac{1}{2}G\nabla^T U = \vec{T}'.$$

That is

$$\begin{array}{rcl} \dot{\vec{\omega}}' & = & J^{-1}\vec{T}' - J^{-1}(\vec{\omega}' \times J\vec{\omega}') - \frac{1}{2}J^{-1}G\nabla^T U \\ \dot{\mathbf{q}} & = & \frac{1}{2}G^T \vec{\omega}'. \end{array} \tag{21}$$

Consider now the Lyapunov candidate

$$V = \frac{1}{2}\vec{\omega}'^T J\vec{\omega}' + U(\mathbf{q}) \tag{22}$$

wich can be made positive definite by proper choice of U. The time derivative of V is then given by

$$\begin{aligned} \dot{V} &= \vec{\omega}'^T J\dot{\vec{\omega}}' + \dot{\mathbf{q}}^T \nabla^T U \\ &= \vec{\omega}'^T J(J^{-1}\vec{T}' - J^{-1}(\vec{\omega}' \times J\vec{\omega}') - \frac{1}{2}J^{-1}G\nabla^T U) + (\frac{1}{2}G^T \vec{\omega}')^T \nabla^T U \\ &= \vec{\omega}'^T \vec{T}' - \frac{1}{2}\vec{\omega}'^T G\nabla^T U + \frac{1}{2}\vec{\omega}'^T G\nabla^T U \\ &= \vec{\omega}'^T \vec{T}'. \end{aligned} \tag{23}$$

Now take again the dissipative control term $-k_v\vec{\omega}'$ and consider the new term $-\frac{1}{2}G\nabla^T U$ in (21) as an additional proportional control torque.
We see that this is equivalent to considering the dynamics of (18) with the control torque $-k_v\vec{\omega}' - \frac{1}{2}G\nabla^T U$ instead of $-k_v\vec{\omega}'$.

We can summarize our findings as follows; the inertial satellite dynamics

$$\begin{array}{rcl} \dot{\vec{\omega}}' & = & J^{-1}\vec{T}' - J^{-1}(\vec{\omega}' \times J\vec{\omega}') \\ \dot{\mathbf{q}} & = & \frac{1}{2}G^T \vec{\omega}' \end{array} \tag{24}$$

with the control torque $\vec{T}' = -k_v\vec{\omega}' - \frac{1}{2}G\nabla^T U$, together with the Lyapunov function (22) satisfies $\dot{V} = -k_v\vec{\omega}'^T\vec{\omega}' \leq 0$.
Reapplying LaSalle's invariance principle, we see that for a $U(\mathbf{q})$ that is positive definite with respect to $\vec{q}$, we tend to an invariant set that may be reduced to the origin. More precisely

$$(\vec{\omega}', \mathbf{q}) \to \{(\vec{\omega}', \mathbf{q}) : \vec{\omega}' = \vec{0},\ U(\mathbf{q}) = 0\} \qquad \text{as} \qquad t \to \infty.$$
$$(\vec{\omega}', \mathbf{q}) \to \{(\vec{\omega}', \mathbf{q}) : \vec{\omega}' = \vec{0},\ \vec{q} = \vec{0}\} \qquad \text{as} \qquad t \to \infty.$$
$$(\vec{\omega}', \mathbf{q}) \to \{(\vec{\omega}', \mathbf{q}) : \vec{\omega}' = \vec{0},\ \mathbf{q} = (\pm 1\ 0\ 0\ 0)^T\} \qquad \text{as} \qquad t \to \infty.$$

at least if no other equilibrium point remains in (24). This means that this type of control law is able to stabilize the satellite at the origin.

5.2.3 Control Law Examples

With $U(\mathbf{q}) = k_p\vec{q}^T\vec{q}$ we get the proportional control term $-\frac{1}{2}G\nabla^T U = -k_p G(0\ \vec{q}^T)^T = -k_p q_0\vec{q}$, leading to the control torque

$$\vec{T}' = -k_v\vec{\omega}' - k_p q_0\vec{q}. \tag{25}$$

For this law, however, additional equilibria appear for

$$(\vec{\omega}', \mathbf{q}) \ \in\ \{(\vec{\omega}', \mathbf{q}) : \vec{\omega}' = \vec{0},\ \mathbf{q} = (q_0\ \vec{q}^T)^T,\ q_0 = 0, \|\vec{q}\| = 1\},$$

but it is easy to see that these are unstable.

An other valid pair is

$$\vec{T}' = -k_v\vec{\omega}' - k_p\vec{q} \tag{26}$$
$$V = \frac{1}{2}\vec{\omega}'^T J\vec{\omega}' + k_p(\vec{q}^T\vec{q} + (q_0 - 1)^2) \tag{27}$$

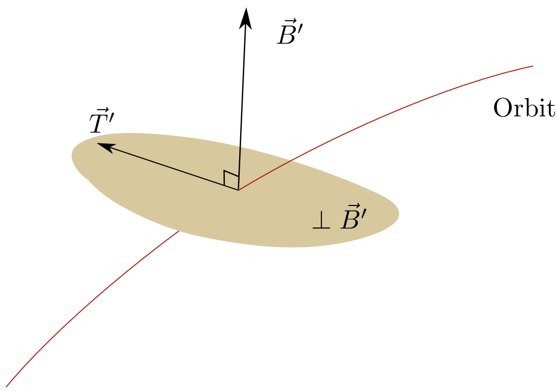
which also satisfies $\dot{V} = -k_v\vec{\omega}'^T\vec{\omega}' \leq 0$. This can be seen noting that $\frac{d}{dt}\vec{q}^T\vec{q} = \frac{d}{dt}(1 - q_0^2) = q_0\vec{q}^T\vec{\omega}'$ and $\frac{d}{dt}(q_0 - 1)^2 = -q_0\vec{q}^T\vec{\omega}' + \vec{q}^T\vec{\omega}'$, that is, $k_p\frac{d}{dt}(\vec{q}^T\vec{q} + (q_0 - 1)^2) = k_p\vec{q}^T\vec{\omega}'$.

5.3 Magnetotorquers Limitation

To generate a control torque $\vec{T}'$, the magnetotorquers create a magnetic field $\vec{M}'$ that interacts with the local Earth's field $\vec{B}' = R^T\vec{B}$. The generated torque is then given by

$$\vec{T}' = \vec{M}' \times \vec{B}'. \tag{28}$$

This means that the control torque is always constrained to the plane perpendicular to $\vec{B}'$, as shown in the following figure.



One can therefore see that one direction is always uncontrollable. However, this uncontrollable direction $\vec{n}' = \frac{\vec{B}'}{\|\vec{B}\|}$ changes with time. In a first approximation, the Earth field expressed in the orbital reference frame has a time-varying and periodical direction

$$\vec{n} = \vec{n}(t) \approx \begin{pmatrix} \sin(\pm\omega_o t) \\ 0 \\ -\cos(\pm\omega_o t) \end{pmatrix} \tag{29}$$

with signs depending on the ORF positive x-direction.
The effect of this constraint can be reformulated as follows: any desired control torque $\vec{T}_d$ is projected in the plane perpendicular to $\vec{n}$, that is

$$\vec{T} = (I_3 - \vec{n}\vec{n}^T)\vec{T}_d.$$

In the body reference frame, this becomes

$$\vec{T}' = P(\mathbf{q}, t)T_d' \qquad \text{with} \qquad P(\mathbf{q}, t) = R^T(I_3 - \vec{n}\vec{n}^T)R. \tag{30}$$

Note that $P(\mathbf{q}, t)$, like $(I_3 - \vec{n}\vec{n}^T)$, is positive semi-definite with eigenvalues $\{1, 1, 0\}$ and is thus non-invertible.

5.4 Averaging

The point of this section is to show that the satellite dynamics can be normalized, and that under certain conditions, the averaging principle can be used. This allows us to write an autonomous system that approximates the solutions of the original non-autonomous one and on which it is easier to solve the control problem.

5.4.1 Projection Matrix

First, note that the complete system is non-autonomous only because of the time dependency of the projection matrix $P(\mathbf{q}, t)$ (30).

Proposition 1. *The mean of the projection matrix $(I_3 - \vec{n}\vec{n}^T)$ with $\|\vec{n}\| = 1$ is symmetric and positive definite if $\vec{n}$ is continuous and satisfies*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|\vec{n} - \vec{\bar{n}}\|^2 > \varepsilon \qquad \forall\ \|\vec{\bar{n}}\| = 1, \vec{\bar{n}}\ \textit{const.}, \varepsilon > 0. \tag{31}$$

(One can interpret this last condition as a "minimal time-variability" requirement)

Proof. First, rewrite the left-hand side of condition (31) in Proposition 1 as

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \|\vec{n}-\vec{\bar{n}}\|^2\ dt\\
=&\lim_{T\to\infty}\frac{1}{T}\int_0^T (\vec{n}-\vec{\bar{n}})^T(\vec{n}-\vec{\bar{n}})\ dt\\
=&\lim_{T\to\infty}\frac{1}{T}\int_0^T \underbrace{\vec{n}^T\vec{n}}_{\equiv 1}-2\vec{\bar{n}}^T\vec{n}+\underbrace{\vec{\bar{n}}^T\vec{\bar{n}}}_{\equiv 1}\ dt\\
=&2-\lim_{T\to\infty}\frac{1}{T}\int_0^T 2\vec{\bar{n}}^T\vec{n}\ dt>\varepsilon\\
\Rightarrow\quad &1-\lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{\bar{n}}^T\vec{n}\ dt>\frac{\varepsilon}{2}\\
\Rightarrow\quad &\lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{\bar{n}}^T\vec{n}\ dt<1 \qquad \forall\ \|\vec{\bar{n}}\|=1,\ \vec{\bar{n}}\text{ const.},\ \varepsilon>0 \qquad (32)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\|\vec{n}-\vec{\bar{n}}\|\leq 2\\
\Rightarrow\quad &\lim_{T\to\infty}\frac{1}{T}\int_0^T (\vec{n}-\vec{\bar{n}})^T(\vec{n}-\vec{\bar{n}})\ dt\leq 4\\
&2-\lim_{T\to\infty}\frac{1}{T}\int_0^T 2\vec{\bar{n}}^T\vec{n}\ dt\leq 4\\
&-\lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{\bar{n}}^T\vec{n}\ dt\leq 1\\
\Rightarrow\quad &\lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{\bar{n}}^T\vec{n}\ dt\geq -1
\end{aligned}$$

For the case $\lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{\bar{n}}^T\vec{n}\ dt=-1$, replace $\vec{\bar{n}}$ by $-\vec{\bar{n}}$ and get 1, which contradicts the assumption (31) because of the first reasoning. Thus

$$-1<\lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{\bar{n}}^T\vec{n}\ dt<1 \qquad (33)$$

Now, consider the time varying matrix $N=\vec{n}\vec{n}^T$ and its mean

$$\bar{N}=\lim_{T\to\infty}\frac{1}{T}\int_0^T N dt.$$

Symmetry of N and $\bar{N}$ is obvious.
For any constant normed vector $\vec{x}$ (i.e. $\vec{x}(t)=\vec{x}\ \forall t,\ \|\vec{x}\|=1$), the quadratic form $\vec{x}^T\bar{N}\vec{x}$ can be written

$$a=\vec{x}^T\bar{N}\vec{x}=\lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{x}^T\vec{n}\vec{n}^T\vec{x}\ dt=\lim_{T\to\infty}\frac{1}{T}\int_0^T (\vec{x}^T\vec{n})^2\ dt.$$

Taking $\vec{x} = \vec{\bar{n}}$ we see that

$$(33) \text{ and } \|\vec{\bar{n}}^T\vec{n}\| \leq 1 \qquad \Rightarrow \qquad 0 \leq a < 1$$

This in turn, implies that $0 \leq \lambda_i\{\bar{N}\} < 1$.
Because $\lambda_i\{I_3 - \bar{N}\} = 1 - \lambda_i\{\bar{N}\}$

$$0 < \lambda_i\{I_3 - \bar{N}\} \leq 1 \qquad \Longrightarrow \qquad \text{positive def.}$$

□

Proposition 2. *The matrix* $\bar{P}(\mathbf{q}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T P(\mathbf{q},t)dt$ *is positive definite and invertible under the same conditions as in Proposition 1*

Proof.

$$\bar{P}(\mathbf{q}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T P(\mathbf{q},t)\ dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T R^T(I_3 - \vec{n}\vec{n}^T)R\ dt = R^T(I_3 - \bar{N})R$$

because R is not time-dependent.

$$R \text{ orthonormal} \qquad \Longrightarrow \qquad \lambda_i\{\bar{P}\} = \lambda_i\{I_3 - \bar{N}\}.$$

That is, $\bar{P}(\mathbf{q})$ is positive definite and thus invertible, as showed in proposition 1.

□

5.4.2 Normalized Non-Inertial Dynamics

Remember that the satellite dynamics expressed in the non-inertial orbital referential, with no inertia wheel and constant ORF rotational speed $\vec{\omega}_o$ is given by

$$\dot{\vec{\omega}}' = J^{-1}\vec{T}' - J^{-1}(\vec{\omega}' \times \vec{u}) + J^{-1}G\Delta[\vec{\omega}_o, \vec{u}]\mathbf{q} - \vec{s}$$

$$\dot{\mathbf{q}} = \tfrac{1}{2}G^T\vec{\omega}'.$$

$$\vec{u} := J(\vec{\omega}' + R^T\vec{\omega}_o)$$
$$\vec{s} := (R^T\vec{\omega}_o) \times \vec{\omega}'.$$

Now consider the change of variables with $j_0 = \lambda_{max}\{J\}$

$$\vec{w}' = \frac{\vec{\omega}'}{\|\vec{\omega}_o\|} \qquad \hat{e}_o = \frac{\vec{\omega}_o}{\|\vec{\omega}_o\|} \qquad \tau = \|\vec{\omega}_o\|t$$
$$\vec{C}' = \frac{\vec{T}'}{T_0} \qquad I = \frac{J}{j_0} \qquad \epsilon = \frac{T_0}{j_0\|\vec{\omega}_o\|^2}$$
$$\cdot = \frac{d}{d\tau}.$$

$\mathbf{q}$ is already dimensionless and of order of magnitude of 1. We can now rewrite the dynamics in those dimensionless variables and obtain

$$\boxed{\begin{array}{c} \dot{\vec{w}}' = \epsilon I^{-1}\vec{C}' - I^{-1}(\vec{w}' \times \vec{u}) + I^{-1}G\Delta[\hat{e}_o, \vec{u}]\mathbf{q} - \vec{s} \\ \\ \dot{\mathbf{q}} = \frac{1}{2}G^T\vec{w}'. \\ \\ \vec{u} := I(\vec{w}' + R^T\hat{e}_o) \\ \vec{s} := (R^T\hat{e}_o) \times \vec{w}'. \end{array}} \tag{34}$$

Note the magnetotoquers' limitation,

$$\vec{C}' = P(\mathbf{q}, t)\vec{C}'_d \ , \tag{35}$$

which makes (34) non-autonomous.

5.4.3 Averaged Dynamics

Using the periodic approximation (29) to construct $P(\mathbf{q}, t)$, considering the non-autonomous system given by (34)-(35) and noting that it is of the form

$$\dot{x} = \epsilon f(x, t) + g(x, t) \ , \tag{36}$$

it appears that we can apply the averaging principle (see [1], section 10.4 and theorem 10.4). Doing this, we approximate the solutions of (36) with the solutions of its average

$$\dot{x} = \epsilon \bar{f}(x) + g(x, t) \qquad\qquad \bar{f}(x) = \frac{1}{T}\int_0^T f(x, t)\ dt.$$

This approximation is then valid for any sufficiently small ϵ. When we apply averaging to the satellite's dynamics, we simply obtain (34) together with

$$\vec{C}' = \bar{P}(\mathbf{q})\vec{C}'_d \tag{37}$$

instead of (35). However, we must be careful before making any conclusion about the stability of the original system based on the analysis of the averaged one, as we will see later.

5.5 Control Law on the Averaged Non-Inertial System

We will now have a look at the normalized, averaged, non-inertial dynamics together with a control law of the type presented in Section 5.2. The system dynamics can be written

$$
\begin{gathered}
\dot{\vec{w}}' = \epsilon I^{-1}\bar{P}(-k_v\vec{w}' - \tfrac{1}{2}\bar{P}^{-1}G\nabla^T U) - I^{-1}(\vec{w}' \times \vec{u}) + I^{-1}G\Delta[\hat{e}_o, \vec{u}]\mathbf{q} - \vec{s} \\
\dot{\mathbf{q}} = \tfrac{1}{2}G^T\vec{w}'. \\
\vec{u} := I(\vec{w}' + R^T\hat{e}_o) \\
\vec{s} := (R^T\hat{e}_o) \times \vec{w}' = \dot{R}^T\hat{e}_o
\end{gathered}
\tag{38}
$$

with $\bar{P} = \bar{P}(\mathbf{q}) = R^T(I_3 - \bar{N})R$ and $\bar{P}^{-1} = R^T(I_3 - \bar{N})^{-1}R$. Note that we have corrected the proportional term with the inverse of the mean projection matrix (more on this later).
Take the following Lyapunov candidate

$$
V = \frac{1}{2}\vec{w}'^T I\vec{w}' + \epsilon U(\mathbf{q}) - \frac{1}{2}\hat{e}_o^T RIR^T\hat{e}_o + \frac{1}{2}\hat{e}_o^T I\hat{e}_o. \tag{39}
$$

(39) is positive definite if $U(\mathbf{q})$ is positive definite in $\vec{q}$ and if $\hat{e}_o$ is the eigenvector to the maximal eigenvalue of I (i.e. axis of maximal inertia), so that $\hat{e}_o^T RIR^T\hat{e}_o \leq \frac{1}{2}\hat{e}_o^T I\hat{e}_o$.
The time derivative of V is then

$$
\begin{aligned}
\dot{V} =& \vec{w}'^T I \dot{\vec{w}}' + \epsilon \dot{\mathbf{q}}^T \nabla^T U - \frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o) \\
=& \vec{w}'^T I \left(\epsilon I^{-1} \bar{P}(-k_v \vec{w}' - \frac{1}{2}\bar{P}^{-1} G \nabla^T U) - I^{-1}(\vec{w}' \times \vec{u}) + I^{-1} G \Delta[\hat{e}_o, \vec{u}]\mathbf{q} - \vec{s} \right) \\
&+ \frac{\epsilon}{2}\vec{w}'^T G \nabla^T U - \frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o) \\
=& -\epsilon k_v \vec{w}'^T \bar{P} \vec{w}' - \cancel{\frac{\epsilon}{2}\vec{w}'^T G \nabla^T U} + \vec{w}'^T G \Delta[\hat{e}_o, \vec{u}]\mathbf{q} - \vec{w}'^T I \dot{R}^T \hat{e}_o \\
&+ \cancel{\frac{\epsilon}{2}\vec{w}'^T G \nabla^T U} - \frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o) \\
=& -\epsilon k_v \vec{w}'^T \bar{P} \vec{w}' + \vec{w}'^T G \Delta[\hat{e}_o, \vec{u}]\mathbf{q} - \vec{w}'^T I \dot{R}^T \hat{e}_o - \frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o) \\
=& -\epsilon k_v \vec{w}'^T \bar{P} \vec{w}' + \vec{w}'^T G \Delta[\hat{e}_o, I(\vec{w}' + R^T \hat{e}_o)]\mathbf{q} - \vec{w}'^T I \dot{R}^T \hat{e}_o - \frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o) \\
=& -\epsilon k_v \vec{w}'^T \bar{P} \vec{w}' + \vec{w}'^T \left(G \Delta[\hat{e}_o, I\vec{w}']\mathbf{q} + G \Delta[\hat{e}_o, I R^T \hat{e}_o]\mathbf{q} \right) \\
&- \vec{w}'^T I \dot{R}^T \hat{e}_o - \frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o) \\
=& -\epsilon k_v \vec{w}'^T \bar{P} \vec{w}' + \dot{\mathbf{q}}^T \left(2\Delta[\hat{e}_o, I\vec{w}']\mathbf{q} + 2\Delta[\hat{e}_o, I R^T \hat{e}_o]\mathbf{q} \right) \\
&- \vec{w}'^T I \dot{R}^T \hat{e}_o - \frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o) \\
=& -\epsilon k_v \vec{w}'^T \bar{P} \vec{w}' + \dot{\mathbf{q}}^T \frac{\partial}{\partial \mathbf{q}} \left(\vec{w}'^T I R^T \hat{e}_o \right) + \cancel{\underbrace{\frac{1}{2}\dot{\mathbf{q}}^T \frac{\partial}{\partial \mathbf{q}} \left(\hat{e}_o^T R I R^T \hat{e}_o \right)}_{\frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o)}} \\
&- \vec{w}'^T I \dot{R}^T \hat{e}_o - \cancel{\frac{d}{dt}(\frac{1}{2}\hat{e}_o^T R I R^T \hat{e}_o)} \\
=& -\epsilon k_v \vec{w}'^T \bar{P} \vec{w}' + \dot{\mathbf{q}}^T \frac{\partial}{\partial \mathbf{q}} \left(\vec{w}'^T I R^T \hat{e}_o \right) - \vec{w}'^T I \dot{R}^T \hat{e}_o
\end{aligned}
$$

And because

$$
\vec{w}' \neq \vec{w}'(\mathbf{q}) \text{ and } R^T = R^T(\mathbf{q}) \quad \Rightarrow \quad \dot{\mathbf{q}}^T \frac{\partial}{\partial \mathbf{q}} \left(\vec{w}'^T I R^T \hat{e}_o \right) = \vec{w}'^T I \dot{R}^T \hat{e}_o
$$

we have

$$
\dot{V} = -\epsilon k_v \vec{w}'^T \bar{P} \vec{w}' \ . \tag{40}
$$

Hence, we again have the same class of control terms as in Section 5.2.2 that implies a semi-negative definite time derivative of a positive definite Lyapunov candidate (39), under the condition that $\hat{e}_o$ is the eigenvector associated to the maximal eigenvalue of I. Control law examples of Section 5.2.3 are still valid here.

5.5.1 Invariant Set

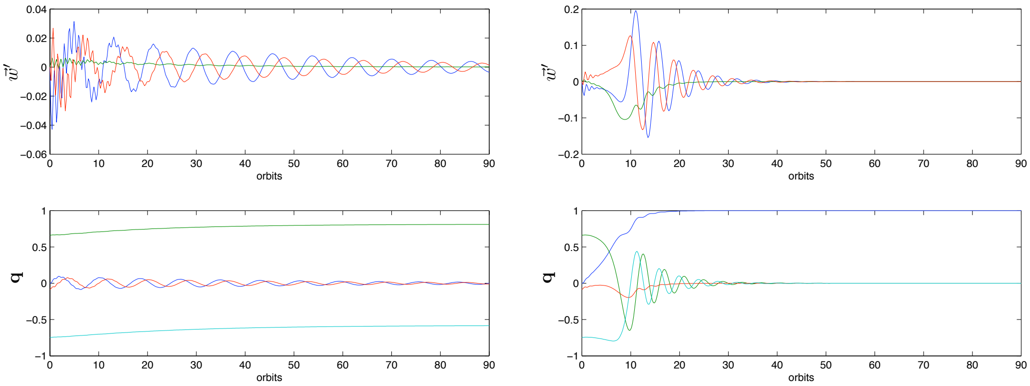
Because (40) is only semi-negative definite, we again have to apply LaSalle's invariance principle before making any convergence conclusion. Actually, (39) and (40) ensure that $(\vec{w}', \mathbf{q})$ will tend to the largest invariant set, but this set may be larger and more difficult to identify than in Section 5.2.2. In fact, it appears that for very small ϵ, the equation

$$0 = \epsilon \bar{f}(x) + g(x, t)$$

seems to have other solutions than the trivial one $(\vec{w}', \mathbf{q}) = (\vec{0}, (1\ 0\ 0\ 0)^T)$ for both control laws proposed in Section 5.2.2. This is certainly due to the fact that the terms

$$-\frac{\epsilon}{2} G \nabla^T U \qquad \text{and} \qquad G \Delta[\hat{e}_o, \vec{u}]\mathbf{q}$$

in system (38) may cancel each other out for quaternion values in $\{\mathbf{q}_{eq}\}$ different from $(1\ 0\ 0\ 0)^T$, making every point $(\vec{w}', \mathbf{q}) \in \{\vec{0}, \mathbf{q} \in \{\mathbf{q}_{eq}\}\}$ an equilibrium point. Moreover, it appears in simulation that (at least) some of those new equilibria are asymptotically stable.
The following figure shows such an equilibrium point ($\epsilon = 0.1$) that disappears for ϵ big enough.



$k_v = k_p = 0.5 \quad \epsilon = 0.1$ $\qquad$ $k_v = k_p = 0.5 \quad \epsilon = 0.5$

Note that this can lead to a problem, since ϵ must be small enough for the averaging principle to apply; however, simulations show that there is a safe interval for ϵ to be chosen in (more on this in Section 5.8.1).

5.5.2 Local Exponential Stability

The stability analysis made so far allows us to conclude that under certain conditions, the averaged system is globally asymptotically stable and it rendered the development of a meaningful control strategy possible. Unfortunately, asymptotic stability of the averaged system is not enough to ensure stability of the original system. By theorem 10.4 in [1], exponential stability is required.
Because no quadratic Lyapunov function with $\dot{V} \leq -k\|x\|$ could be found yet, we propose to check local exponential stability through linearization of (34)-(37), (29) and with either control law of Section 5.2.3 (they are equivalent near

the origin). With

$$I = \begin{pmatrix} 0.7190 & 0.0313 & -0.0044 \\ 0.0313 & 1.0000 & -0.0294 \\ -0.0044 & -0.0294 & 0.8998 \end{pmatrix} \qquad k_v = k_p = 0.5 \quad \epsilon = 0.5$$

and using code from `linearization_no_wheel.m` we get the following vector of eigenvalues for the linear system matrix A

$$\vec{\lambda}(A) = \begin{pmatrix} -0.1183 + 1.1114i \\ -0.1183 - 1.1114i \\ -0.0545 + 0.3166i \\ -0.0545 - 0.3166i \\ -0.1091 + 0.3540i \\ -0.1091 - 0.3540i \\ 0 \end{pmatrix}.$$

All eigenvalues have negative a real part, except for one, which again corresponds to the uncontrollable length of the quaternion vector (i.e. the coordinate algebraic constraint). This means that the linear system is stable, implying local exponential stability for the original system (for the numerical values given and for ϵ sufficiently small).

5.6 Practical Control Law

In the control laws proposed until now, the inverse of the averaged projection matrix $\bar{P}^{-1}(\mathbf{q}) = R^T(I_3 - \bar{N})^{-1}R$ was always involved. But while

$$(I_3 - \bar{N}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

under the approximation (29), this may be quite wrong in practice. However, it appears in simulation, that a control law

$$\vec{C}' = -k_v \vec{w}' - k_p \vec{q}$$

or

$$\vec{C}' = -k_v \vec{w}' - k_p q_0 \vec{q}$$

(that is without $\bar{P}^{-1}$ in front of the proportional term) works as well. This could only be verified experimentally, because no global Lyapunov function could be found for these simplified control laws. In fact, the stability of the system seems very robust to any pre-multiplication by a positive matrix of the control torque; but again, this could not be shown analytically. The results in [2] could not be verified as the Lyapunov function proposed seems in fact to have an indefinite time derivative.

5.7 Issue of Maximal Rejectable Perturbation

The averaging theorem tells us that the averaged system (34)+(37) has solutions close to those of the original system (34)+(35) for all $\epsilon < \epsilon^*$, with ϵ^* a small

constant that has to be estimated experimentally. In fact, we are less interested in the limit for ϵ so that solutions of both systems are close than in a limit of ϵ for which system (34)+(35) remains stable; indeed, it appears that for $\epsilon > \epsilon^{**}$, the projection error of the control term makes the system unstable.
Simulations of (34)+(35) showed that an $\epsilon < \epsilon^{**}$ with

$$\epsilon^{**} \approx 5$$

made the system remains stable. And because ϵ was defined as $\epsilon = \frac{T_0}{j_0 \|\vec{\omega}_o\|^2}$, we can estimate the maximal magnitude of the control torque with

$$T_{max} = \epsilon^{**} j_0 \|\vec{\omega}_o\|^2.$$

Taking $j_0 = 1.5\ 10^{-3} kg\ m^2$ and $\|\vec{\omega}_o\| = \frac{1}{90\ min} \approx 1.85\ 10^{-4}\ s^{-1}$ we get

$$T_{max} \approx 2.6\ 10^{-10} Nm.$$

Unfortunately, this is very much smaller than the estimated maximal disturbance torques estimated in [10], which are

$$3.7\ 10^{-7} Nm \text{ at } 400\ km \qquad \text{and} \qquad 4.4\ 10^{-8} Nm \text{ at } 1000\ km.$$

This means (and could be confirmed by simulation), that the perturbations from the environment cannot be rejected.

Note: Intuitively, one sees from the normalized dynamics, that $\|\vec{\omega}_o\|$ gives the order of magnitude (o.o.m.) for the rotation velocities involved in the problem and $\frac{1}{\|\vec{\omega}_o\|}$ gives the time unit suited for describing those phenomena. That is, the torques involved (incl. disturbances) should be of such magnitude, so that the rotation velocity is changed by an o.o.m. of $\|\vec{\omega}_o\|$ on a time interval of an o.o.m. of $\|\vec{\omega}_o\|^{-1}$; because the control torque is averaged on a similar time interval. And this is the case when ϵ is of an o.o.m. of 1.
Put in an other way, due to the magnetotorquers limitation, it is also intuitively clear that in general, stabilization from some initial conditions cannot occur in less than one orbit time (it takes one orbit time for reaching all control directions at least once).

5.8 Some Simulation Results

In this section, we present some simulation results related to the theory developed earlier.

5.8.1 Averaging Validation

Here we want to check if the averaging of the system is valid and to what extent. To do so, we simulate both the averaged normalized and the non-autonomous normalized systems (i.e. with $\bar{P}(\mathbf{q})$ and $P(\mathbf{q}, \tau)$) starting with the same initial conditions. The control law used here is $\vec{C}' = -k_v \vec{w}' - k_p \vec{q}$ (the one described in Section 5.6). Figures 5 and 6 show simulations for varying ϵ.
As we can see, for values of ϵ below 1, both systems have very similar solutions. For ϵ between 1 and 5, trajectories look somewhat different but both systems

stabilize on equivalent time intervals. However, for ϵ larger than 5, the averaged system continues to stabilize on decreasing time intervals for increasing ϵ, while the non-autonomous original system explodes. The code used in these simulations is `compare_av_nonav.m`.

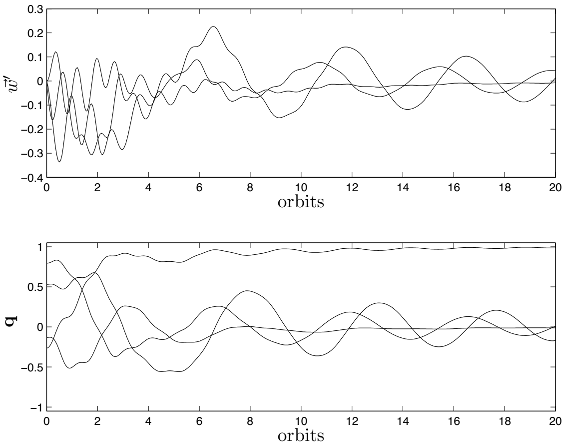
5.8.2 Simulations on the Non-Normalized System

Finally, we want to have a look at the behavior of the original non-normalized, non-autonomous system (17) with a control law of the type $-k_v\vec{\omega}' - k_p\vec{q}$ applied on it. Note that now, k_v and k_p are no more of o.o.m. 1.
In order to avoid unnecessary energy waste, we generate a magnetic field that is perpendicular to the local Earth field (that is, we predict the effect of the projection matrix). Let be $\vec{T}'_d = -k_v\vec{\omega}' - k_p\vec{q}$ the desired control torque and $\vec{T}' = \vec{M}' \times \vec{B}'$ the actual control torque. Then we have to generate the field

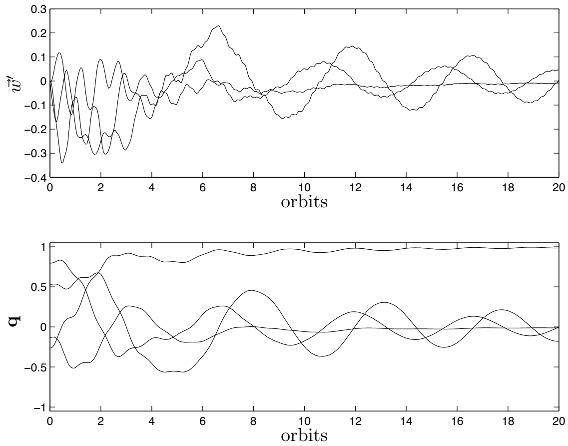
$$\vec{M}' = \frac{1}{\|\vec{B}'\|^2}\,\vec{B}' \times \vec{T}'_d = \frac{1}{\vec{B}'^T\vec{B}'}\,\vec{B}' \times (-k_v\vec{\omega}' - k_p\vec{q})$$

to create the proper projected torque, without loosing power (since $\|\vec{a} \times \vec{b}\| = \sin(\theta)\|\vec{a}\|\|\vec{b}\|$ is maximal for $\vec{a} \perp \vec{b}$).

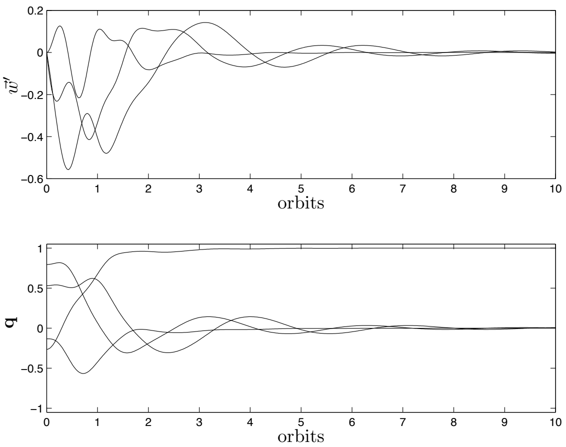
Figures 7 to 9 show some simulations done with the code in folder `Simul-CompleteSystem/`. The file `main.m` is the one that has to be launched and the torques are computed in the file `Torque.m`.



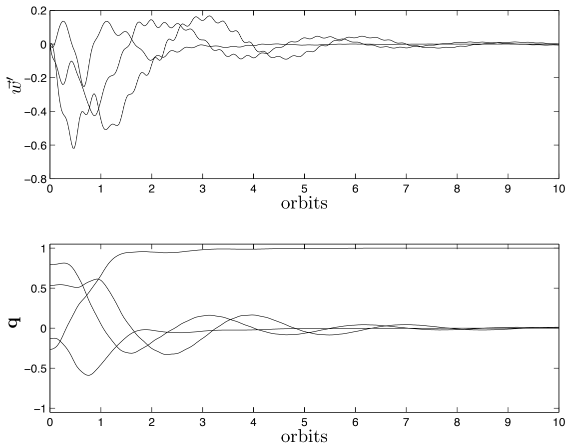
Averaged dynamics, $\epsilon = 0.1$ $k_p = k_v = 1$



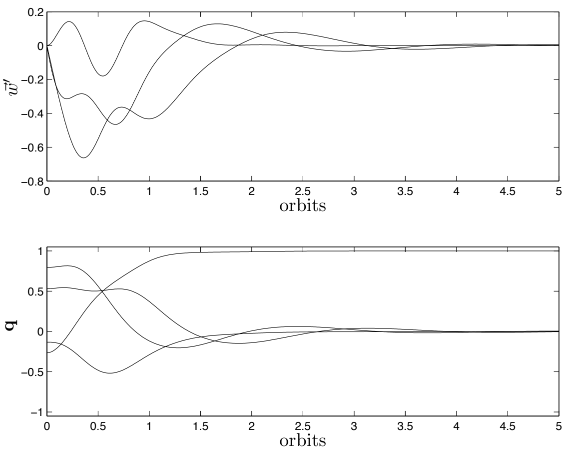
Original dynamics, $\epsilon = 0.1$ $k_p = k_v = 1$



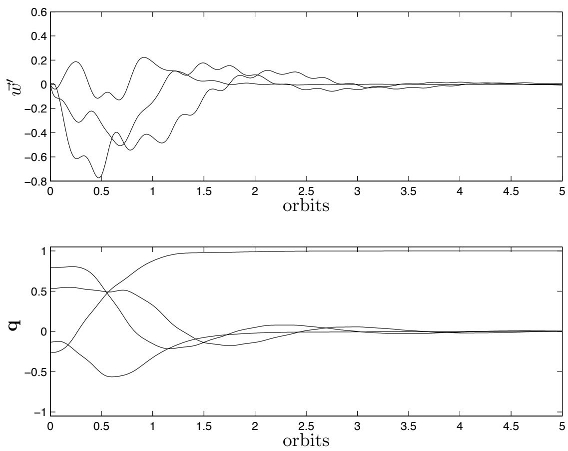
Averaged dynamics, $\epsilon = 0.5$ $k_p = k_v = 1$



Original dynamics, $\epsilon = 0.5$ $k_p = k_v = 1$

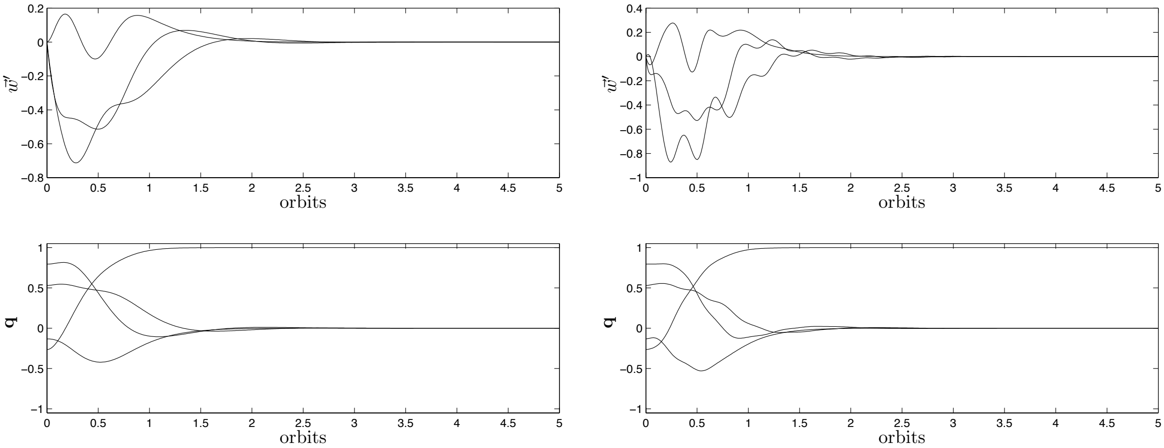


Averaged dynamics, $\epsilon = 1$ $k_p = k_v = 1$

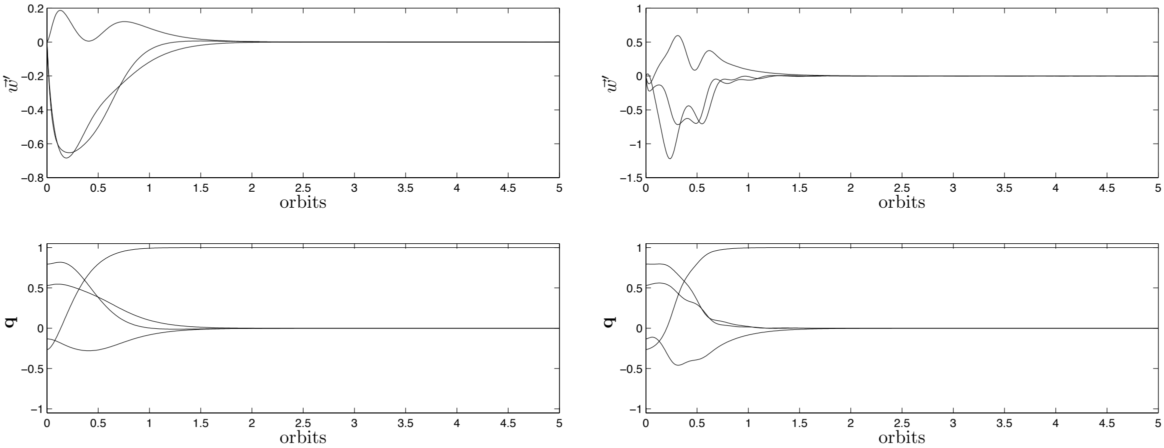


Original dynamics, $\epsilon = 1$ $k_p = k_v = 1$

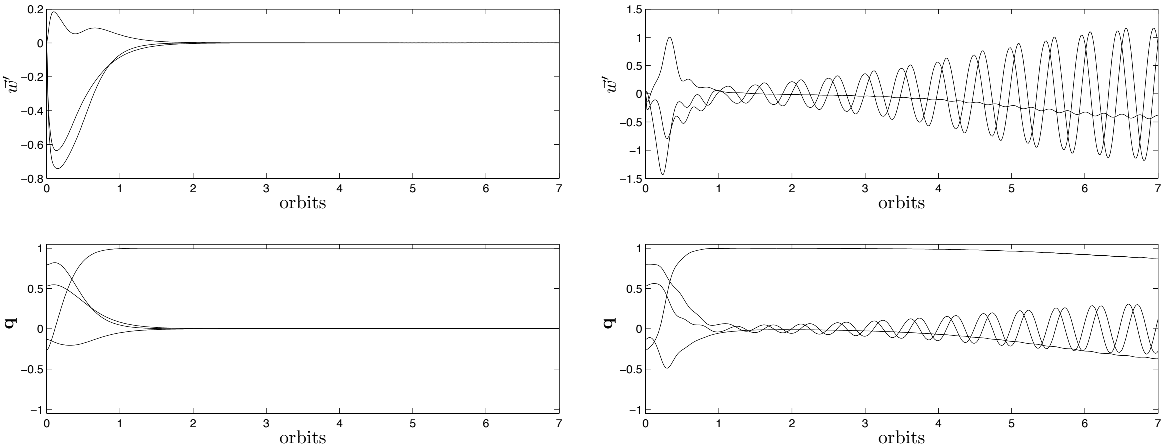
Figure 5: Comparison between averaged normalized and original (non-autonomous) normalized dynamics with varying ϵ.



Averaged dynamics, $\epsilon = 2 \;\; k_p = k_v = 1$ Original dynamics, $\epsilon = 2 \;\; k_p = k_v = 1$

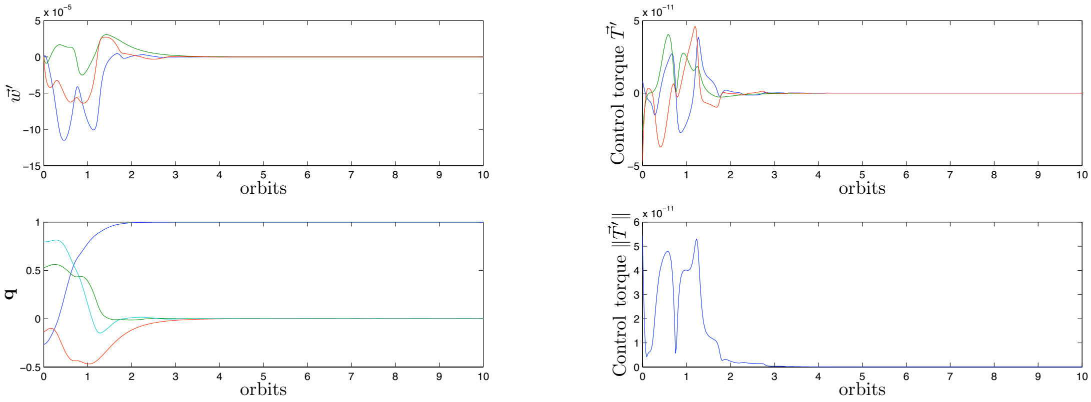


Averaged dynamics, $\epsilon = 5 \;\; k_p = k_v = 1$ Original dynamics, $\epsilon = 5 \;\; k_p = k_v = 1$

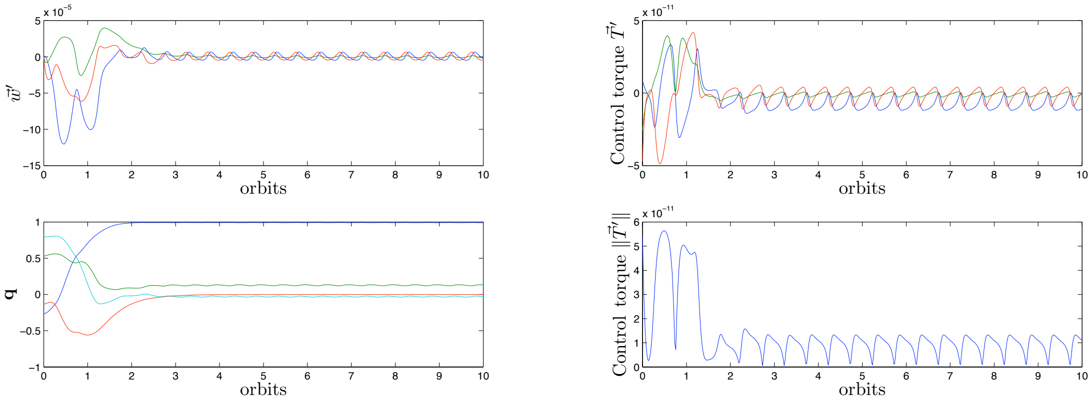


Averaged dynamics, $\epsilon = 10 \;\; k_p = k_v = 1$ Original dynamics, $\epsilon = 10 \;\; k_p = k_v = 1$

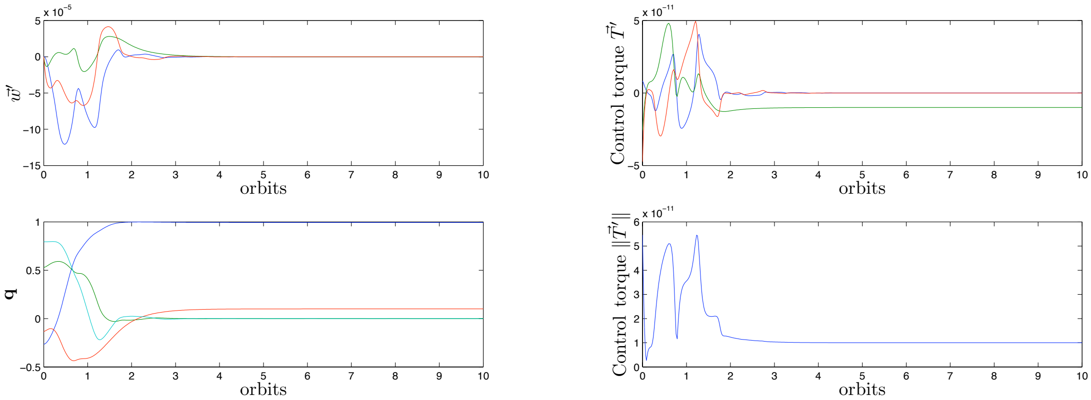
Figure 6: Comparison between averaged normalized and original (non-autonomous) normalized dynamics with varying ϵ.



a) Disturbance torque $\vec{T}_{dist} = 10^{-11}(0\ 0\ 0)^T\ Nm$, $k_v = 10^{-6}$, $k_p = 10^{-10}$

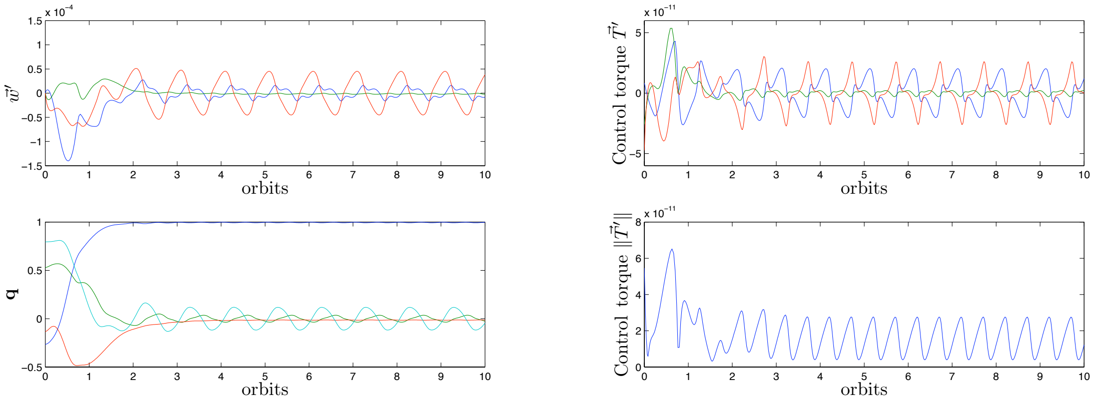


b) Disturbance torque $\vec{T}_{dist} = 10^{-11}(1\ 0\ 0)^T\ Nm$, $k_v = 10^{-6}$, $k_p = 10^{-10}$

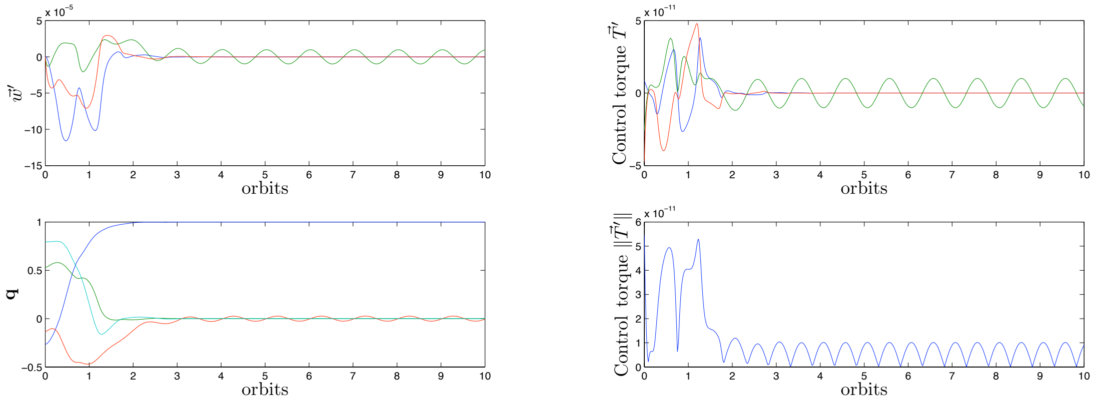


c) Disturbance torque $\vec{T}_{dist} = 10^{-11}(0\ 1\ 0)^T\ Nm$, $k_v = 10^{-6}$, $k_p = 10^{-10}$

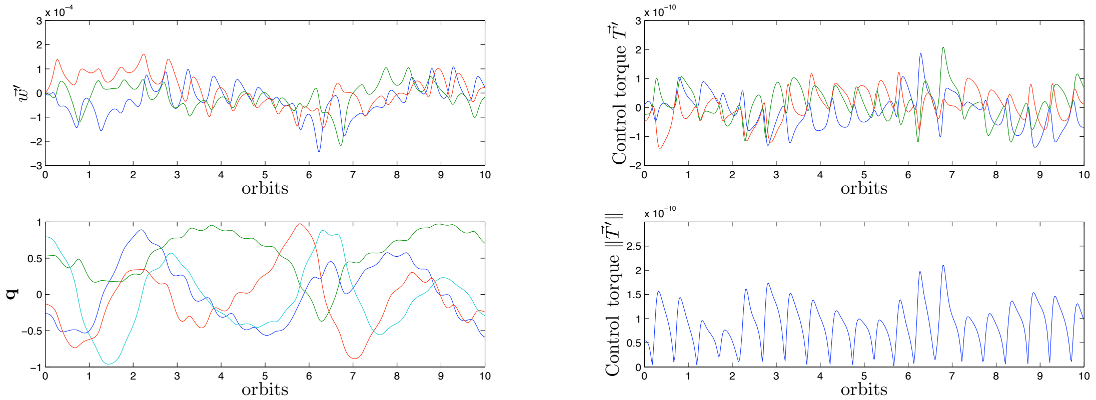
Figure 7: Original system with various disturbance torques. The disturbance torque is given in the orbital reference frame. a) No disturbance and therefore no statism. b) Disturbance torque in the x direction, this direction is not always controllable making oscillations appear. c) Disturbance torque in the y direction, this direction is always controllable, therefore, we observe a constant statism.



a) Disturbance torque $\vec{T}_{dist} = \cos(\omega_o t) \cdot 10^{-11} (0\ 0\ 1)^T\ Nm$, $k_v = 10^{-6}$, $k_p = 10^{-10}$

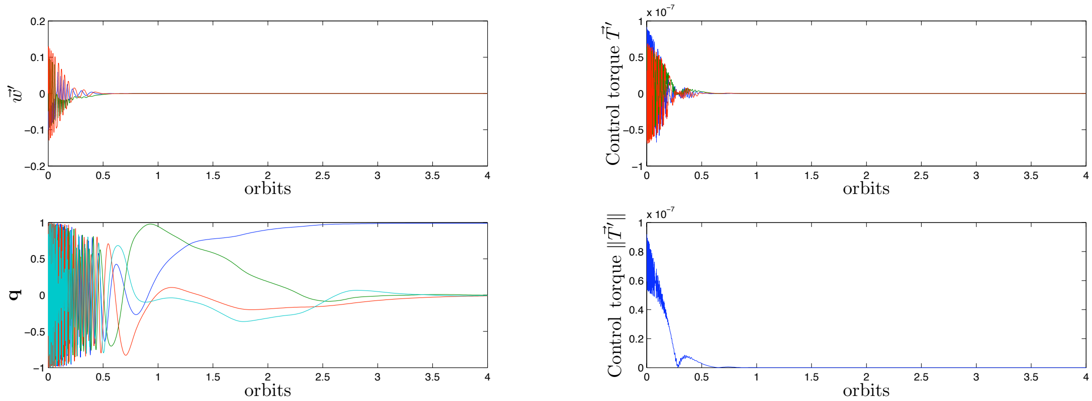


b) Disturbance torque $\vec{T}_{dist} = \cos(\omega_o t) \cdot 10^{-11} (0\ 1\ 0)^T\ Nm$, $k_v = 10^{-6}$, $k_p = 10^{-10}$

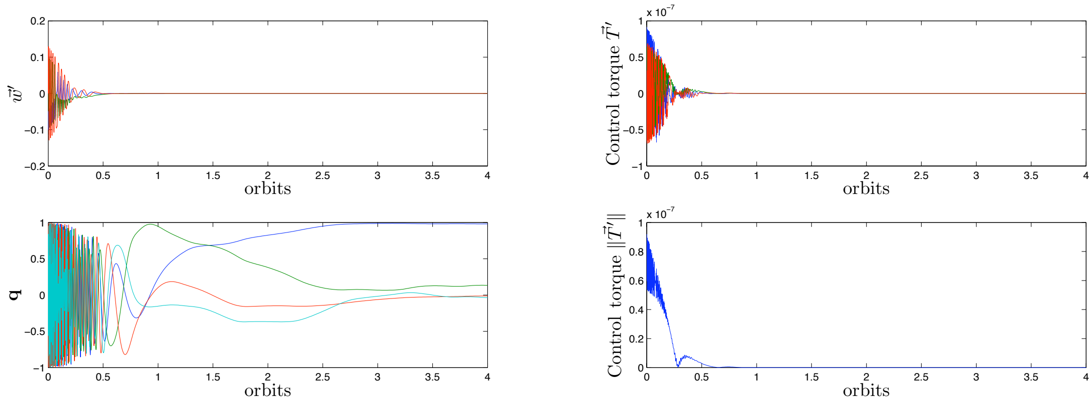


c) Disturbance torque $\vec{T}_{dist} = 8 \cdot 10^{-11} (1\ 0\ 0)^T\ Nm$, $k_v = 10^{-6}$, $k_p = 10^{-10}$

Figure 8: Original system with various disturbance torques. The disturbance torque is given in the orbital reference frame. a) Periodical disturbance along ORF x axis b) Periodical disturbance along ORF y axis; note that the perturbation rejection is better because the y direction is always controllable. c) Disturbance torque to big for the controller to reject them.



a) $\vec{\omega}'_{t=0} = 0.1 \cdot (-1\ 1\ 0)^T\ \frac{rad}{s}$, $\vec{T}_{dist} = \vec{0}$, $k_v = 10^{-6}$, $k_p = 10^{-10}$



b) $\vec{\omega}'_{t=0} = 0.1 \cdot (-1\ 1\ 0)^T\ \frac{rad}{s}$, $\vec{T}_{dist} = 1 \cdot 10^{-11}(1\ 0\ 0)^T\ Nm$, $k_v = 10^{-6}$, $k_p = 10^{-10}$

Figure 9: Original system with high initial rotational velocity. The disturbance torque is given in the orbital reference frame. a) No disturbance torque. b) with disturbance torque along x ORF axis. We see that initial velocity is not a problem.

6 Conclusions

During this project, good theoretical and intuitive comprehension of the system has been acquired.
The use of one inertia wheel was seen to be problematic, as it would have to be placed along the only axis that is always controllable with the magnetotorquers in nominal attitude and as it would penalize controllability on the other axes if spinning too fast.
Full magnetic actuation was found to be theoretically feasible, but unfortunately, was also found to be unable to reject the amount of environmental predicted perturbations.

Finally, I would like to thank all the people from the "Laboratoire d'Automatique", in particular my supervisors Philippe Muellhaupt and Sebastien Gros, for their help and valuable advice.

A Quaternions

A.1 Fundamentals

Relation (41), together with associativity and distributivity is all what we will use to derive the basic practical applications for quaternions.

$$\boxed{i^2 = j^2 = k^2 = ijk = -1} \tag{41}$$

By left- and right-multiplication in the above equation, we can write

$$\begin{array}{cc} \multicolumn{2}{c}{i\ ijk = -jk = -i} \\ \multicolumn{2}{c}{ijk\ k = -ij = -k} \\ j\ jk = -k = ji & ij\ j = -i = kj \\ i\ ij = -j = ik & ji\ i = -j = -ki \end{array}$$

This shows the product is *non commutative* and gives the basic multiplication rules:

$$\boxed{\begin{array}{cc} ij = k & ji = -k \\ jk = i & kj = -i \\ ki = j & ik = -j \end{array}} \tag{42}$$

A.2 Notations and Definitions

A quaternion q is a set of four parameters, a real value q_0 and three imaginary values $q_1 i$, $q_2 j$, $q_3 k$ with $q_1, q_2, q_3 \in \mathbb{R}$; it may be written

$$q = q_0 + q_1 i + q_2 j + q_3 k.$$

However, this notation proves itself to be very unpractical. We will therefore use two different notations:

- The quaternion q as a pair of real value and vectorial imaginary value
  $q = (q_0, \vec{q})$ $\qquad \mathrm{Re}\{q\} = q_0 \qquad \vec{\mathrm{Im}}\{q\} = \vec{q} = (q_1\ q_2\ q_3)^T$
- A column vector of four parameters
  $\mathbf{q} = (q_0\ q_1\ q_2\ q_3)^T$

The conjugate $\bar{q}$ of q is defined as

$$\bar{q} = (q_0, -\vec{q})$$

and it's norm (a nonnegative real value) as

$$|q| = |\mathbf{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

The product of two quaternions written as pairs, as described in the next section will be noted with $\circ$.

A.3 Quaternion Product

From the rules given in (42), we may write the product of q with p.

$$
\begin{array}{rcccccccl}
\multicolumn{9}{c}{(q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k) \;=} \\
 & p_0 q_0 & + & q_0 p_1\ i & + & q_0 p_2\ j & + & q_0 p_3\ k & \\
+ & q_1 p_0\ i & + & q_1 p_1\ ii & + & q_1 p_2\ ij & + & q_1 p_3\ ik & \\
+ & q_2 p_0\ j & + & q_2 p_1\ ji & + & q_2 p_2\ jj & + & q_2 p_3\ jk & \\
+ & q_3 p_0\ k & + & q_3 p_1\ ki & + & q_3 p_2\ kj & + & q_3 p_3\ kk & =
\end{array}
$$

$$
\begin{array}{rcccccccl}
 & p_0 q_0 & - & q_1 p_1 & - & q_2 p_2 & - & q_3 p_3 & \\
+ & (q_1 p_0 & + & q_0 p_1 & + & q_2 p_3 & - & q_3 p_2)\ i & \\
+ & (q_2 p_0 & + & q_0 p_2 & + & q_3 p_1 & - & q_1 p_3)\ j & \\
+ & (q_3 p_0 & + & q_0 p_3 & + & q_1 p_2 & - & q_2 p_1)\ k &
\end{array}
$$

$$q \circ p = (p_0 q_0 - \vec{p} \cdot \vec{q}, q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p}). \tag{43}$$

From (43) it turns out that

$$q \circ \bar{q} = \bar{q} \circ q = (|q|^2, \vec{0}) = |q|^2 \tag{44}$$

and if q is normed ($|q| = 1$)

$$q \circ \bar{q} = \bar{q} \circ q = (1, \vec{0}) = Id. \tag{45}$$

In (43) we also see that

$$\overline{q \circ p} = \bar{p} \circ \bar{q} \tag{46}$$

that is

$$|q \circ p|^2 = (q \circ p) \circ (\overline{q \circ p}) = q \circ \underbrace{p \circ \bar{p}}_{|p|^2} \circ \bar{q} = |p|^2 (q \circ \bar{q}) = |q|^2 |p|^2$$

$$|q \circ p| = |q||p|. \tag{47}$$

A.4 Quaternions and Spatial Rotations

First, note the following relations

$$(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$$

$$\sin^2 \frac{\varphi}{2} = \frac{1 - \cos\varphi}{2} \qquad \cos^2 \frac{\varphi}{2} = \frac{1 + \cos\varphi}{2}.$$

From now on, q will generally represent a normed quaternion ($|q| = 1$) involved in a rotation. Let's now place a vector $\vec{x} \in \mathbb{R}^3$ in the imaginary part of a quaternion x and see what happens with it in the following relation

$$x' = \bar{q} \circ x \circ q \qquad x = (0, \vec{x}) \qquad q = (q_0, \vec{q}).$$

Using (43)

$$
\begin{aligned}
x' &= (\vec{q}\cdot\vec{x},\ q_0\vec{x}-\vec{q}\times\vec{x})\circ q \\
&= (\underbrace{(\vec{q}\cdot\vec{x})q_0-(q_0\vec{x}-\vec{q}\times\vec{x})\cdot\vec{q}}_{\mathrm{Re}\{x'\}},\ \underbrace{(\vec{q}\cdot\vec{x})\vec{q}+q_0(q_0\vec{x}-\vec{q}\times\vec{x})+(q_0\vec{x}-\vec{q}\times\vec{x})\times\vec{q}}_{\vec{\mathrm{Im}}\{x'\}})
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Re}\{x'\} &= (\vec{q}\cdot\vec{x})q_0 - q_0(\vec{x}\cdot\vec{q}) - (\vec{q}\times\vec{x})\cdot\vec{q} = 0 \\
&\Rightarrow\ x' = (0,\vec{x}'), \\
\vec{\mathrm{Im}}\{x'\} &= \vec{x}' \\
&= (\vec{q}\cdot\vec{x})\vec{q} + q_0^2\vec{x} - q_0(\vec{q}\times\vec{x}) + q_0(\vec{x}\times\vec{q}) - (\vec{q}\times\vec{x})\times\vec{q} \\
&= (\vec{q}\cdot\vec{x})\vec{q} + q_0^2\vec{x} + 2q_0(\vec{x}\times\vec{q}) - (\vec{q}\times\vec{x})\times\vec{q} \\
&= (\vec{q}\cdot\vec{x})\vec{q} + q_0^2\vec{x} + 2q_0(\vec{x}\times\vec{q}) - (\vec{q}\cdot\vec{q})\vec{x} + (\vec{x}\cdot\vec{q})\vec{q} \\
&= 2(\vec{q}\cdot\vec{x})\vec{q} + q_0^2\vec{x} + 2q_0(\vec{x}\times\vec{q}) - (\vec{q}\cdot\vec{q})\vec{x}.
\end{aligned}
$$

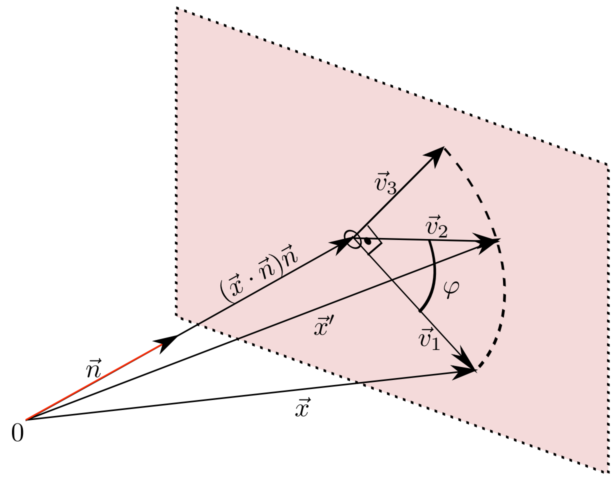
A valid normed quaternion ($|q| = \sqrt{(q_0^2+q_1^2+q_2^2+q_3^2)} = 1$) would be

$$
q = (q_0,\vec{q}) = (\cos\frac{\varphi}{2},\ \sin\frac{\varphi}{2}\vec{n}) \qquad\qquad |\vec{n}| = 1.
$$

In this case, $\vec{x}'$ becomes

$$
\begin{aligned}
\vec{x}' &= 2\sin^2\frac{\varphi}{2}(\vec{n}\cdot\vec{x})\vec{n} + \cos^2\frac{\varphi}{2}\vec{x} + 2\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}(\vec{x}\times\vec{n}) - \sin^2\frac{\varphi}{2}\vec{x} \\
&= (1-\cos\varphi)(\vec{n}\cdot\vec{x})\vec{n} + \cos\varphi\ \vec{x} + \sin\varphi\ (\vec{x}\times\vec{n}).
\end{aligned}
$$

This last relation is the formula for a rotation by an angle φ around a normed axis vector $\vec{n}$, as can be shown with the following figure as follows:



$$
\begin{aligned}
\vec{v}_2 &= \cos\varphi\ \vec{v}_1 + \sin\varphi\ \vec{v}_3 \\
\vec{v}_1 &= \vec{x} - (\vec{x}\cdot\vec{n})\vec{n} \\
\vec{v}_3 &= \vec{v}_1 \times \vec{n} \\
&= (\vec{x} - (\vec{x}\cdot\vec{n})\vec{n}) \times \vec{n} \\
&= (\vec{x}\times\vec{n}) - (\vec{x}\cdot\vec{n})\underbrace{\vec{n}\times\vec{n}}_{\vec{0}} \\
\Rightarrow\quad \vec{v}_2 &= \cos\varphi\ (\vec{x} - (\vec{x}\cdot\vec{n})\vec{n}) + \sin\varphi\ (\vec{x}\times\vec{n})
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}' &= (\vec{x}\cdot\vec{n})\vec{n} + \vec{v}_2 \\
&= (\vec{x}\cdot\vec{n})\vec{n} + \cos\varphi\ (\vec{x} - (\vec{x}\cdot\vec{n})\vec{n}) + \sin\varphi\ (\vec{x}\times\vec{n}) \\
&= (1-\cos\varphi)(\vec{n}\cdot\vec{x})\vec{n} + \cos\varphi\ \vec{x} + \sin\varphi\ (\vec{x}\times\vec{n}).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
x' &= \bar{q}\circ x\circ q \\
q\circ x'\circ\bar{q} &= \underbrace{q\circ\bar{q}}_{(1,\vec{0})}\circ x\circ\underbrace{q\circ\bar{q}}_{(1,\vec{0})}.
\end{aligned}
$$

Thus we have the relations for the rotation and its inverse

$$
\boxed{x' = \bar{q}\circ x\circ q \qquad x = q\circ x'\circ\bar{q}}. \tag{48}
$$

A.5 Quaternions and Rotation Velocity

We will now derive the relation between the rotational velocity vector and the quaternion time derivative. $\vec{x}'$ is any constant vector within the *body (rotating) reference frame* and $\vec{x}$ is the same vector in the *fixed reference frame*. As seen before, both vectors can be put in relation with

$$
x = q\circ x'\circ\bar{q} \qquad x' = \bar{q}\circ x\circ q.
$$

Applying the time derivative to $x = (0,\vec{x})$, with $x' = (0,\vec{x}')$ and $\dot{\vec{x}}' = \vec{0}$, we get

$$
\begin{aligned}
\dot{x} &= \dot{q}\circ x'\circ\bar{q} + q\circ x'\circ\dot{\bar{q}} \\
\dot{x} &= \dot{q}\circ\bar{q}\circ x\circ\underbrace{q\circ\bar{q}}_{Id} + \underbrace{q\circ\bar{q}}_{Id}\circ x\circ q\circ\dot{\bar{q}} \\
\dot{x} &= \dot{q}\circ\bar{q}\circ x + x\circ q\circ\dot{\bar{q}}
\end{aligned} \tag{49}
$$

and from (43)

$$
\begin{aligned}
\dot{q}\circ\bar{q} &= (\underbrace{\dot{q}_0 q_0 + \dot{\vec{q}}\cdot\vec{q}}_{\circledast}, -\dot{q}_0\vec{q} + q_0\dot{\vec{q}} - \dot{\vec{q}}\times\vec{q}) \\
\circledast &= q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 = \mathbf{q}\cdot\dot{\mathbf{q}} = 0
\end{aligned}
$$

because $|\mathbf{q}| = 1$. That is

$$\dot{q} \circ \bar{q} = (0, \vec{\nu}) \qquad \text{and similarly} \qquad \bar{q} \circ \dot{q} = (0, \vec{-\nu}). \tag{50}$$

A.5.1 Rotation Velocity in Fixed Reference Frame ω

From (49) and (50) and using (43) we have

$$\begin{aligned} \dot{x} &= (0, \vec{\nu}) \circ x - x \circ (0, \vec{-\nu}) \\ \dot{\vec{x}} &= \vec{\nu} \times \vec{x} - \vec{x} \times \vec{\nu} = 2\vec{\nu} \times \vec{x} \end{aligned}$$

and from (47)

$$|\dot{\vec{x}}| = |2\vec{\nu}||\vec{x}| \qquad \Rightarrow \qquad \vec{\nu} \perp \vec{x}$$

If $\vec{x}$ undergoes a pure rotation, we know that

$$\dot{\vec{x}} = \vec{\omega} \times \vec{x} \qquad \text{and} \qquad \vec{\omega} \perp \vec{x}$$

thus

$$\boxed{\omega = (0, \vec{\omega}) = 2(0, \vec{\nu}) = 2\dot{q} \circ \bar{q}}. \tag{51}$$

And right-multiplication by q

$$\omega \circ q = 2\dot{q} \circ \underbrace{\bar{q} \circ q}_{Id} \quad \Rightarrow \quad \omega \circ q = 2\dot{q}$$

$$\boxed{\dot{q} = \tfrac{1}{2}\omega \circ q}. \tag{52}$$

A.5.2 Rotation Velocity in Body Reference Frame ω'

$$\begin{aligned} &\omega' = \bar{q} \circ \omega \circ q \qquad \text{with} \qquad \omega = 2\dot{q} \circ \bar{q} \\ \Rightarrow \quad &\omega' = 2\bar{q} \circ \dot{q} \circ \underbrace{\bar{q} \circ q}_{Id} \end{aligned}$$

$$\boxed{\omega' = 2\bar{q} \circ \dot{q}}. \tag{53}$$

And left-multiplication by q

$$q \circ \omega' = 2\underbrace{q \circ \bar{q}}_{Id} \circ \dot{q} = 2\dot{q}$$

$$\boxed{\dot{q} = \tfrac{1}{2} q \circ \omega'}. \tag{54}$$

A.5.3 Matrix-Product Notation for ω

From

$$\omega = 2\dot{q} \circ \bar{q}$$

and using (43)

$$\begin{aligned}
\vec{\omega} = \vec{\mathrm{Im}}\{2\dot{q} \circ \bar{q}\} &= 2(-\dot{q}_0\vec{q} + q_0\dot{\vec{q}} - \dot{\vec{q}} \times \vec{q}) \\
&= 2 \underbrace{\begin{pmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix}}_{E} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix}
\end{aligned}$$

$$\vec{\omega} = 2E\dot{\mathbf{q}}.$$

Changing the sign and inverting the cross product allows to make an other identification

$$\vec{\omega} = -2(-q_0\dot{\vec{q}} + \dot{q}_0\vec{q} - \vec{q} \times \dot{\vec{q}})$$

$$\vec{\omega} = -2\dot{E}\mathbf{q}.$$

So the rotation velocity vector in the *fixed reference* frame can be written as

$$\boxed{\vec{\omega} = 2E\dot{\mathbf{q}} = -2\dot{E}\mathbf{q}}. \tag{55}$$

And from

$$\dot{q} = \frac{1}{2}\omega \circ q \qquad \omega = (0, \vec{\omega}) \;\Rightarrow\; \omega_0 = 0$$

one can similarly find

$$\dot{\mathbf{q}} = \frac{1}{2} \begin{pmatrix} (-\vec{\omega} \cdot \vec{q}) \\ (q_0\vec{\omega} + \vec{\omega} \times \vec{q}) \end{pmatrix} = \frac{1}{2}E^T\vec{\omega}$$

$$\boxed{\dot{\mathbf{q}} = \tfrac{1}{2}E^T\vec{\omega}}. \tag{56}$$

A.5.4 Matrix-Product Notation for ω'

From

$$\omega' = 2\bar{q} \circ \dot{q}$$

and using (43)

$$\vec{\omega'} = \vec{\mathrm{Im}}\{2\bar{q} \circ \dot{q}\} = 2(q_0\dot{\vec{q}} - \dot{q}_0\vec{q} - \vec{q} \times \dot{\vec{q}})$$

$$= 2 \underbrace{\begin{pmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{pmatrix}}_{G} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix}$$

$$\vec{\omega'} = 2G\dot{\mathbf{q}}.$$

Changing the sign and inverting the cross product allows to make an other identification

$$\vec{\omega'} = -2(\dot{q}_0\vec{q} - q_0\dot{\vec{q}} - \dot{\vec{q}} \times \vec{q})$$

$$\vec{\omega'} = -2\dot{G}\mathbf{q}.$$

So the rotation velocity vector in the *body reference* frame can be written as

$$\boxed{\vec{\omega'} = 2G\dot{\mathbf{q}} = -2\dot{G}\mathbf{q}}. \tag{57}$$

And from

$$\dot{q} = \frac{1}{2}q \circ \omega' \qquad \omega' = (0, \vec{\omega'}) \;\Rightarrow\; \omega'_0 = 0$$

one can similarly find

$$\dot{\mathbf{q}} = \frac{1}{2}\begin{pmatrix} (-\vec{q} \cdot \vec{\omega'}) \\ (q_0\vec{\omega'} + \vec{q} \times \vec{\omega'}) \end{pmatrix} = \frac{1}{2}G^T\vec{\omega'}$$

$$\boxed{\dot{\mathbf{q}} = \tfrac{1}{2}G^T\vec{\omega'}}. \tag{58}$$

A.5.5 Rotation Matrix R

We already have

$$\vec{\omega} = 2E\dot{\mathbf{q}} = -2\dot{E}\mathbf{q} \qquad \vec{\omega'} = 2G\dot{\mathbf{q}} = -2\dot{G}\mathbf{q}$$

$$\dot{\mathbf{q}} = \frac{1}{2}E^T\vec{\omega} \qquad \dot{\mathbf{q}} = \frac{1}{2}G^T\vec{\omega'}$$

So we can write

$$\begin{aligned} \vec{\omega} &= 2E\dot{\mathbf{q}} \\ &= 2E(\frac{1}{2}E^T\vec{\omega}) \\ &= EE^T\vec{\omega} \\ &\Rightarrow \boxed{EE^T = Id}. \end{aligned} \qquad \begin{aligned} \vec{\omega'} &= 2G\dot{\mathbf{q}} \\ &= 2G(\frac{1}{2}G^T\vec{\omega'}) \\ &= GG^T\vec{\omega'} \\ &\Rightarrow \boxed{GG^T = Id}. \end{aligned}$$

And by mixing both sides

$$\vec{\omega'} = 2G\dot{\mathbf{q}} = 2G(\frac{1}{2}E^T\vec{\omega}) = GE^T\vec{\omega}$$
$$\vec{\omega} = 2E\dot{\mathbf{q}} = 2E(\frac{1}{2}G^T\vec{\omega'}) = EG^T\vec{\omega'}.$$

We shall now remember that $\vec{\omega}$ is a vector in the *fixed reference frame* and that $\vec{\omega}'$ is the same vector in the *body reference frame*, that is $\vec{\omega} = R\vec{\omega'}$. By comparing with the previous two results, we find

$$\boxed{R = EG^T} \qquad \text{and} \qquad \boxed{R^{-1} = R^T = GE^T}. \tag{59}$$

A.5.6 $E\mathbf{p}$ and $G\mathbf{p}$

From the identifications made in sections A.5.3 and A.5.4, we can see that the general meaning the product of E and G with any quaternion $\mathbf{p}$ is

$$E\mathbf{p} = \vec{\mathrm{Im}}\{p \circ \bar{q}\} \qquad\qquad G\mathbf{p} = \vec{\mathrm{Im}}\{\bar{q} \circ p\}. \tag{60}$$

And from

$$q \circ \bar{q} = \bar{q} \circ q = (|q|, \vec{0}) = (1, \vec{0})$$

it follows

$$\boxed{E\mathbf{q} = \vec{0}} \qquad\qquad \boxed{G\mathbf{q} = \vec{0}}.$$

A.5.7 One Last Relation

For any $\vec{v}$ and due to associativity

$$\underbrace{(0, \vec{\omega'})}_{2\bar{q}\circ\dot{q}} \circ (0, \vec{v}) = (-\vec{\omega'} \cdot \vec{v}, \vec{\omega'} \times \vec{v})$$

$$\begin{aligned}
&= 2\bar{q} \circ \dot{q} \circ v \\
&= 2(q_0\dot{q}_0 + \vec{q} \cdot \dot{\vec{q}}, q_0\dot{\vec{q}} - \dot{q}_0\vec{q} - \vec{q} \times \dot{\vec{q}}) \circ v = 2\bar{q} \circ (\dot{q}_0 v_0 - \dot{\vec{q}} \cdot \vec{v}, \dot{q}_0\vec{v} + v_0\dot{\vec{q}} + \dot{\vec{q}} \times \vec{v}) \\
&\equiv 2 \left(\begin{array}{cccc} q_0 & q_1 & q_2 & q_3 \\ \hline -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{array}\right) \left(\begin{array}{c|ccc} \dot{q}_0 & -\dot{q}_1 & -\dot{q}_2 & -\dot{q}_3 \\ \dot{q}_1 & \dot{q}_0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_2 & \dot{q}_3 & \dot{q}_0 & -\dot{q}_1 \\ \dot{q}_3 & -\dot{q}_2 & \dot{q}_1 & \dot{q}_0 \end{array}\right) \left(\begin{array}{c} 0 \\ \hline v_1 \\ v_2 \\ v_3 \end{array}\right) \\
&= 2 \begin{pmatrix} \mathbf{q}^T \\ G \end{pmatrix} \begin{pmatrix} \dot{\mathbf{q}} & \dot{G}^T \end{pmatrix} \begin{pmatrix} 0 \\ \vec{v} \end{pmatrix} = \begin{pmatrix} -\vec{\omega'} \cdot \vec{v} \\ \vec{\omega'} \times \vec{v} \end{pmatrix} \\
&\Rightarrow \; 2G\dot{G}^T\vec{v} = \Omega'\vec{v} = \vec{\omega'} \times \vec{v}.
\end{aligned}$$

Comparing with (57), we conclude that

$$\boxed{\Omega' = 2G\dot{G}^T = -2\dot{G}G^T \qquad \text{and} \qquad \Omega'\vec{v} = \vec{\omega'} \times \vec{v}}. \tag{61}$$

A.5.8 Relations Summary

The following table summaries the developed relations. q is always a normed quaternion, that is $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

Quaternion notation		Matrix notation	
Fixed ref	*Body ref*	*Fixed ref*	*Body ref*
$x = q \circ x' \circ \bar{q}$	$x' = \bar{q} \circ x \circ q$	$\vec{x} = R\vec{x}'$ $R = EG^T$	$\vec{x}' = R^T \vec{x}$ $R^T = R^{-1} = GE^T$
$\omega = (0, \vec{\omega}) = 2\dot{q} \circ \bar{q}$	$\omega' = (0, \vec{\omega}') = 2\bar{q} \circ \dot{q}$	$\vec{\omega} = 2E\dot{\mathbf{q}} = -2\dot{E}\mathbf{q}$	$\vec{\omega}' = 2G\dot{\mathbf{q}} = -2\dot{G}\mathbf{q}$
$\dot{q} = \frac{1}{2}\omega \circ q$	$\dot{q} = \frac{1}{2}q \circ \omega'$	$\dot{\mathbf{q}} = \frac{1}{2}E^T\vec{\omega}$	$\dot{\mathbf{q}} = \frac{1}{2}G^T\vec{\omega}'$
		$EE^T = Id$	$GG^T = Id$
$q \circ \bar{q} = \bar{q} \circ q = (\lvert q \rvert, \vec{0})$		$E\mathbf{q} = \vec{0}$	$G\mathbf{q} = \vec{0}$
	$(0, \vec{\omega}) \circ (0, \vec{v}) =$ $(-\vec{\omega}' \cdot \vec{v}, \vec{\omega}' \times \vec{v})$		$\Omega' = 2G\dot{G}^T$ $= -2\dot{G}G^T$ $\Omega'\vec{v} = \vec{\omega}' \times \vec{v}$

$$E = \begin{pmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \qquad G = \begin{pmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{pmatrix}$$

A.6 Rigid Body Rotational Dynamics

We now will have a look at the dynamics of a freely rotating rigid body to which a momentum $\vec{T}'$ is applied. Translation of the body will not be discussed (it can be decoupled from the dynamics of rotation and is fairly easy). We will also consider a potential free system, so that the Lagrangian resumes to the rotational kinetic energy only

$$L = E_{\text{rot}} = \frac{1}{2}\vec{\omega}'^T J \vec{\omega}'. \tag{62}$$

Using the quaternion $\mathbf{q}$ as coordinates and with the constraint $C = \mathbf{q}^T\mathbf{q} = 1$, Lagrangian dynamics gives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{F_q} + \lambda \frac{\partial C}{\partial \mathbf{q}}. \tag{63}$$

$\mathbf{F_q}$ is the 4-vector of generalized forces which will be expressed in term of applied torque later. λ is the Lagrangian multiplier used to satisfy the constraint C.

A.6.1 Derivatives of L

Note the following reminder

$$\frac{\partial A\mathbf{x}}{\partial \mathbf{x}} = A$$
$$\frac{\partial \mathbf{a}^T\mathbf{x}}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}^T\mathbf{a}}{\partial \mathbf{x}} = \mathbf{a}$$
$$\frac{\partial \mathbf{x}^T A\mathbf{x}}{\partial \mathbf{x}} = (A^T + A)\mathbf{x} \overset{\text{if } A=A^T}{=} 2A\mathbf{x}$$
(written as column vectors)
$$(AB)^T = B^T A^T \ .$$

We will now derive each term of the left side of (63). First, let us rewrite L in two different ways

$$L = \frac{1}{2}\vec{\omega}'^T J\vec{\omega}' = 2(G\dot{\mathbf{q}})^T J(G\dot{\mathbf{q}}) = 2(\dot{G}\mathbf{q})^T J(\dot{G}\mathbf{q})$$

and grouping around J

$$L = \frac{1}{2}\vec{\omega}'^T J\vec{\omega}' = 2\dot{\mathbf{q}}^T(G^T JG)\dot{\mathbf{q}} = 2\mathbf{q}^T(\dot{G}^T J\dot{G})\mathbf{q}.$$

Because J is symmetric, $(G^T JG)$ and $(\dot{G}^T J\dot{G})$ are also symmetric. So we have

$$\frac{\partial L}{\partial \mathbf{q}} = 4\dot{G}^T J\dot{G}\mathbf{q} = 2\dot{G}^T J\underbrace{(2\dot{G}\mathbf{q})}_{-\vec{\omega}'} = -2\dot{G}^T J\vec{\omega}', \tag{64}$$

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = 4G^T JG\dot{\mathbf{q}} = 2G^T J\underbrace{(2G\dot{\mathbf{q}})}_{\vec{\omega}'} = 2G^T J\vec{\omega}'$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{d}{dt}(2G^T J\vec{\omega}') = 2\dot{G}^T J\vec{\omega}' + 2G^T J\dot{\vec{\omega}}'. \tag{65}$$

A.6.2 Generalized Forces

A way to find the generalized force $\mathbf{F_c}$ relative to the coordinates $\mathbf{c}$ is to identify it in

$$\delta W = \mathbf{F_c} \cdot \delta\mathbf{c}.$$

(A simple example is the case of a pure translation $\delta\vec{x}$ of a particle, on which a force $\vec{F}$ is applied. The work is then $\delta W = \mathbf{F}_{\vec{x}} \cdot \delta\vec{x} = \vec{F} \cdot \delta\vec{x}$. So the generalized force $\mathbf{F}_{\vec{x}}$ is simply $\vec{F}$ in this case.)

For a rotation of a rigid body by an angle $\delta\varphi$ around an axis $\vec{n}$ with an applied torque $\vec{T}'$, the work can be written as

$$\delta W = (\vec{n} \cdot \vec{T}')\delta\varphi \qquad |\vec{n}| = 1. \tag{66}$$

This small attitude change can be represented on one side as a small variation δq of the coordinate quaternion q and, on the other side, as a rotation quaternion q_δ operating from the current attitude represented by q (i.e. a composition). That is

$$q + \delta q = q \circ q_\delta$$
$$|q| = 1 \qquad |q_\delta| = 1 \qquad |\delta q| \ll 1.$$

We do not need to consider the fact that the variation δq has to preserve the norm of q, because it will automatically be satisfied by introducing a constraint in the Lagrange formulation.

On one side we can write

$$q + \delta q = q \circ q_\delta$$
$$\underbrace{\bar{q} \circ q}_{(1,\vec{0})} + \bar{q} \circ \delta q = \underbrace{\underbrace{\bar{q} \circ q}_{(1,\vec{0})} \circ q_\delta}_{q_\delta}$$

$$\Rightarrow \;\; q_\delta = (1, \vec{0}) + \bar{q} \circ \delta q. \tag{67}$$

On the other side

$$q_\delta = (\cos \frac{\delta\varphi}{2}, \; \sin \frac{\delta\varphi}{2} \; \vec{n}).$$

Looking at the imaginary part

$$\vec{\mathrm{Im}}\{q_\delta\} = \vec{\mathrm{Im}}\{\bar{q} \circ \delta q\} = \sin \frac{\delta\varphi}{2} \; \vec{n} \approx \frac{\delta\varphi}{2} \; \vec{n}$$

comparing with (66)

$$\Rightarrow \;\; \delta W = 2 \; \vec{\mathrm{Im}}\{\bar{q} \circ \delta q\} \cdot \vec{T}'$$

and from (60)

$$\vec{\mathrm{Im}}\{\bar{q} \circ \delta q\} = G\delta\mathbf{q}$$
$$\Rightarrow \;\; \delta W = 2(G\delta\mathbf{q}) \cdot \vec{T}' = 2\vec{T}'^T(G\delta\mathbf{q}) = 2(G^T\vec{T}')^T\delta\mathbf{q} = \underbrace{2(G^T\vec{T}')}_{\mathbf{F_q}} \cdot \delta\mathbf{q}$$

$$\Rightarrow \;\; \boxed{\mathbf{F_q} = 2G^T\vec{T}'}. \tag{68}$$

A.6.3 Dynamics

We have now everything to write the dynamics

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} &= \mathbf{F_q} + \lambda \frac{\partial C}{\partial \mathbf{q}} \\ 4\dot{G}^T J\vec{\omega}' + 2G^T J\dot{\vec{\omega}}' &= 2G^T \vec{T}' + \lambda \mathbf{q}.\end{aligned}$$

Left-multiplying by G

$$\begin{aligned}\underbrace{4G\dot{G}^T}_{2\Omega'} J\vec{\omega}' + 2\underbrace{GG^T}_{Id} J\dot{\vec{\omega}}' &= 2\underbrace{GG^T}_{Id} \vec{T}' + \lambda \underbrace{G\mathbf{q}}_{\vec{0}} \\ \Omega' J\vec{\omega}' + J\dot{\vec{\omega}}' &= \vec{T}' \\ \vec{\omega}' \times J\vec{\omega}' + J\dot{\vec{\omega}}' &= \vec{T}' \\ J\dot{\vec{\omega}}' &= \vec{T}' - \vec{\omega}' \times J\vec{\omega}'.\end{aligned}$$

This last relation is nothing else than the Euler equation of motion for rotating body. Together with (58) we obtain the complete dynamics

$$\boxed{\begin{array}{lcl}\dot{\vec{\omega}}' & = & J^{-1}\vec{T}' - J^{-1}(\vec{\omega}' \times J\vec{\omega}') \\ \dot{\mathbf{q}} & = & \frac{1}{2}G^T\vec{\omega}'.\end{array}} \tag{69}$$

B Derivatives and Quaternions

B.1 Quadratic Form Derivative by a Quaternion

In order to be able to derive the Lagrangian by the components of $\mathbf{q}$ in the non-inertial quaternion model, we need to perform things like

$$\frac{\partial(\vec{v}^T R \vec{w})}{\partial \mathbf{q}} \quad , \qquad \frac{\partial(\vec{v}^T R^T \vec{w})}{\partial \mathbf{q}}$$

and also

$$\frac{\partial(\vec{u}^T R J R^T \vec{u})}{\partial \mathbf{q}}.$$

But because $R = EG^T$ and

$$E = \begin{pmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \qquad G = \begin{pmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{pmatrix}$$

the matrix of the quadratic form to be derived is not constant in $\mathbf{q}$ anymore as it was the case in the inertial model. This implies that these operations are no more trivial. However, thanks to the particular form of the dependance of R in the components of $\mathbf{q}$, higher order tensors can be avoided, as shown in the following.

B.1.1 "Single R" Quadratic Form

By computing the quadratic form and taking the partial derivatives we get (placing them in a column vector)

$$\frac{\partial(\vec{v}^T R \vec{w})}{\partial \mathbf{q}} = \left(\frac{\partial(\vec{v}^T R \vec{w})}{\partial \mathbf{q}_i} \right)_i =$$

$$2 \begin{pmatrix} w_1 v_1 q_0 + w_1 v_2 q_3 - w_1 v_3 q_2 - w_2 v_1 q_3 + w_2 v_2 q_0 + w_2 v_3 q_1 + w_3 v_1 q_2 - w_3 v_2 q_1 + w_3 v_3 q_0 \\ w_1 v_1 q_1 + w_1 v_2 q_2 + w_1 v_3 q_3 + w_2 v_1 q_2 - w_2 v_2 q_1 + w_2 v_3 q_0 + w_3 v_1 q_3 - w_3 v_2 q_0 - w_3 v_3 q_1 \\ -w_1 v_1 q_2 + w_1 v_2 q_1 - w_1 v_3 q_0 + w_2 v_1 q_1 + w_2 v_2 q_2 + w_2 v_3 q_3 + w_3 v_1 q_0 + w_3 v_2 q_3 - w_3 v_3 q_2 \\ -w_1 v_1 q_3 + w_1 v_2 q_0 + w_1 v_3 q_1 - w_2 v_1 q_0 - w_2 v_2 q_3 + w_2 v_3 q_2 + w_3 v_1 q_1 + w_3 v_2 q_2 + w_3 v_3 q_3 \end{pmatrix}.$$

The vector obtained is quite ugly but one can see that it is linear in $\mathbf{q}$, it can thus be rewritten in a matrix-vector product:

$$\underbrace{2 \begin{pmatrix} v_1 w_1 + v_2 w_2 + v_3 w_3 & v_3 w_2 - v_2 w_3 & -v_3 w_1 + v_1 w_3 & v_2 w_1 - v_1 w_2 \\ v_3 w_2 - v_2 w_3 & v_1 w_1 - v_2 w_2 - v_3 w_3 & v_1 w_2 + v_2 w_1 & v_1 w_3 + v_3 w_1 \\ -v_3 w_1 + v_1 w_3 & v_1 w_2 + v_2 w_1 & v_2 w_2 - v_1 w_1 - v_3 w_3 & v_2 w_3 + v_3 w_2 \\ v_2 w_1 - v_1 w_2 & v_1 w_3 + v_3 w_1 & v_2 w_3 + v_3 w_2 & v_3 w_3 - v_1 w_1 - v_2 w_2 \end{pmatrix}}_{\Delta[\vec{v},\vec{w}]} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

By careful inspection of $\Delta[\vec{v},\vec{w}]$, we can identify a structure in the matrix that allows a compact notation

$$\Delta[\vec{v},\vec{w}] = \begin{pmatrix} \vec{w}\cdot\vec{v} & (\vec{w}\times\vec{v})^T \\ \vec{w}\times\vec{v} & \vec{w}\vec{v}^T + \vec{v}\vec{w}^T - \vec{w}\cdot\vec{v}\ I_3 \end{pmatrix}. \tag{70}$$

That is

$$\frac{\partial(\vec{v}^T R\vec{w})}{\partial\mathbf{q}} = 2\Delta[\vec{v},\vec{w}]\mathbf{q} \tag{71}$$

And because $\vec{v}^T R^T\vec{w} = \vec{w}^T R\vec{v}$ we also have

$$\frac{\partial(\vec{v}^T R^T\vec{w})}{\partial\mathbf{q}} = 2\Delta[\vec{w},\vec{v}]\mathbf{q} \tag{72}$$

B.1.2 "Double R" Quadratic Form

We are now interested in the derivative of a quadratic form involving RJR^T, that is, with the $\mathbf{q}$ dependent matrix R appearing twice. J is an inertia matrix, therefore, $J = J^T$. This time, the vectors on the left ant on the right are the same, lets say $\vec{u}$.

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial\mathbf{q}}\left(\vec{u}^T RJR^T\vec{u}\right) &= \frac{1}{2}\left(\vec{u}^T\frac{\partial R}{\partial\mathbf{q}_i}JR^T\vec{u}\right)_i + \frac{1}{2}\left(\vec{u}^T RJ\frac{\partial R^T}{\partial\mathbf{q}_i}\vec{u}\right)_i \\ &= \left(\vec{u}^T\frac{\partial R}{\partial\mathbf{q}_i}JR^T\vec{u}\right)_i.\end{aligned}$$

Hence

$$\frac{1}{2}\frac{\partial}{\partial\mathbf{q}}\left(\vec{u}^T RJR^T\vec{u}\right) = 2\Delta[\vec{u},JR^T\vec{u}]\mathbf{q} \tag{73}$$

B.1.3 Properties

By looking at (70), one may note the following relations

$$\Delta[\vec{v}_1+\vec{v}_2,\vec{w}] = \Delta[\vec{v}_1,\vec{w}] + \Delta[\vec{v}_2,\vec{w}] \tag{74}$$

$$\Delta[\vec{v},\vec{w}_1+\vec{w}_2] = \Delta[\vec{v},\vec{w}_1] + \Delta[\vec{v},\vec{w}_2] \tag{75}$$

$$\Delta\left[\sum_{i=1}^{n}\vec{v}_i, \sum_{j=1}^{m}\vec{w}_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m}\Delta[\vec{v}_i,\vec{w}_j] \tag{76}$$

$$\Delta[\alpha\vec{v},\beta\vec{w}] = \alpha\beta\Delta[\vec{v},\vec{w}] \tag{77}$$

B.2 Time Derivative of $R\vec{w}$ and $R^T\vec{w}$

In computing $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\mathbf{q}}}$ in the non-inertial quaternion model, we see that we need to take the time derivative of expressions of the form $R\vec{w}$ with R the time dependent rotation matrix. Remembering that $R = EG^T$ and verifying that $\dot{E}G^T = E\dot{G}^T$, we may write

$$\begin{aligned}\dot{R} &= \dot{E}G^T + E\dot{G}^T = 2E\dot{G}^T \\ \dot{R}^T &= \dot{G}E^T + G\dot{E}^T = 2G\dot{E}^T.\end{aligned}$$

We can now concentrate on the products $\dot{G}^T\vec{w}$ and $\dot{E}^T\vec{w}$

$$\dot{G}^T\vec{w} = \begin{pmatrix} -\dot{q}_1\, w_1 - \dot{q}_2\, w_2 - \dot{q}_3\, w_3 \\ \dot{q}_0\, w_1 - \dot{q}_3\, w_2 + \dot{q}_2\, w_3 \\ \dot{q}_3\, w_1 + \dot{q}_0\, w_2 - \dot{q}_1\, w_3 \\ -\dot{q}_2\, w_1 + \dot{q}_1\, w_2 + \dot{q}_0\, w_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -w_1 & -w_2 & -w_3 \\ w_1 & 0 & w_3 & -w_2 \\ w_2 & -w_3 & 0 & w_1 \\ w_3 & w_2 & -w_1 & 0 \end{pmatrix}}_{\Gamma[\vec{w}]} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix}$$

$$\dot{E}^T\vec{w} = \begin{pmatrix} -\dot{q}_1\, w_1 - \dot{q}_2\, w_2 - \dot{q}_3\, w_3 \\ \dot{q}_0\, w_1 + \dot{q}_3\, w_2 - \dot{q}_2\, w_3 \\ -\dot{q}_3\, w_1 + \dot{q}_0\, w_2 + \dot{q}_1\, w_3 \\ \dot{q}_2\, w_1 - \dot{q}_1\, w_2 + \dot{q}_0\, w_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -w_1 & -w_2 & -w_3 \\ w_1 & 0 & -w_3 & w_2 \\ w_2 & w_3 & 0 & -w_1 \\ w_3 & -w_2 & w_1 & 0 \end{pmatrix}}_{\bar{\Gamma}[\vec{w}]} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix}.$$

That is $\frac{\partial}{\partial t}(R\vec{w}) = \dot{R}\vec{w} + R\dot{\vec{w}}$ and $\frac{\partial}{\partial t}(R^T\vec{w}) = \dot{R}^T\vec{w} + R^T\dot{\vec{w}}$ can both be computed using

$$\dot{R}\vec{w} = 2E\dot{G}^T\vec{w} = 2E\Gamma[\vec{w}]\dot{\mathbf{q}} = E\Gamma[\vec{w}]G^T\vec{\omega}' \tag{78}$$

$$\dot{R}^T\vec{w} = 2G\dot{E}^T\vec{w} = 2G\bar{\Gamma}[\vec{w}]\dot{\mathbf{q}} = G\bar{\Gamma}[\vec{w}]G^T\vec{\omega}' \tag{79}$$

See next section for a more useful form.

B.3 Time Derivative of R

First note that by identification, one can verify that

$$G^TG = E^TE = I_4 - \mathbf{q}\mathbf{q}^T \tag{80}$$

with I_4 the identity matrix in $\mathbb{R}^4$. Remember also

$$\Omega' = 2G\dot{G}^T = -2\dot{G}G^T \qquad \text{with} \qquad \Omega'\vec{v} = \vec{\omega}' \times \vec{v}$$

and

$$\vec{\omega}' = 2G\dot{\mathbf{q}} = -2\dot{G}\mathbf{q}.$$

Now observe

$$\begin{aligned}
\Omega' R^T &= 2G\dot{G}^T G E^T \\
&= -2\dot{G} G^T G E^T \\
&= -2\dot{G}(I_4 - \mathbf{q}\mathbf{q}^T) E^T \\
&= -2\dot{G} E^T - 2\dot{G}\mathbf{q}\ \underbrace{\mathbf{q}^T E^T}_{(E\mathbf{q})^T = \vec{0}} \\
&= -2\dot{G} E^T = -\dot{R}^T.
\end{aligned}$$

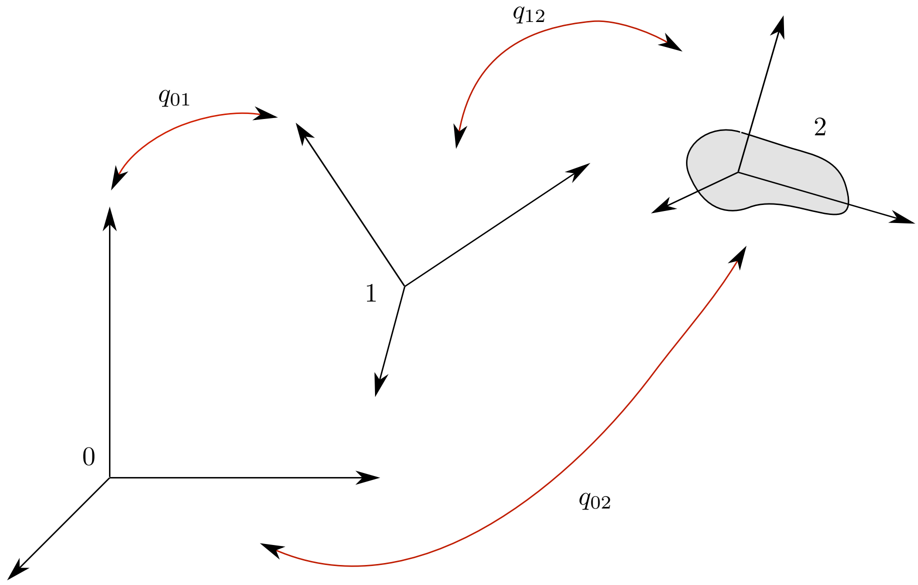
We can finally write

$$\dot{R}^T = -\Omega' R^T \tag{81}$$

$$\dot{R} = -R\Omega'^T = R\Omega'. \tag{82}$$

C Speed Composition

Let be three referentials each designed by 0, 1 and 2. Referential 0 is inertial, referential 1 is a rotating one and 2 is a body fixed referential.
The same vector $\vec{x}$ can be expressed in any of these referentials; when expressed in 0, we will notate it as $\vec{x}^0$, when expressed in 1 it will be noted $\vec{x}^1$ and $\vec{x}^2$ in referential 2. We will also write x^i the quaternion $(0, \vec{x}^i)$.
Moreover, three quaternions are defined: q_{01} describes relative attitude of referential 1 with respect to referential 0, q_{12} describes relative attitude of referential 2 with respect to referential 1 and q_{02} describes relative attitude of referential 2 with respect to referential 0.



So we may write

$$x^0 = q_{01} \circ x^1 \circ \bar{q}_{01} \qquad x^1 = q_{12} \circ x^2 \circ \bar{q}_{12} \qquad x^0 = q_{02} \circ x^2 \circ \bar{q}_{02}$$

and by substitution

$$x^0 = q_{01} \circ x^1 \circ \bar{q}_{01} = q_{01} \circ q_{12} \circ x^2 \circ \bar{q}_{12} \circ \bar{q}_{01} = (q_{01} \circ q_{12}) \circ x^2 \circ (\overline{q_{01} \circ q_{12}})$$

we can identify q_{02}

$$q_{02} = q_{01} \circ q_{12}. \tag{83}$$

Noting $\omega_{ij}^j = (0, \vec{\omega}_{ij}^j)$ the rotation velocity of the reference frame j relative to frame i expressed in the frame j and remembering that $\omega_{ij}^j = 2\bar{q}_{ij} \circ \dot{q}_{ij}$, we may write

$$
\begin{aligned}
\omega_{02}^2 &= 2\bar{q}_{02} \circ \dot{q}_{02} \\
&= 2(\bar{q}_{12} \circ \bar{q}_{01}) \circ (\dot{q}_{01} \circ q_{12} + q_{01} \circ \dot{q}_{12}) \\
&= 2\bar{q}_{12} \circ \bar{q}_{01} \circ \dot{q}_{01} \circ q_{12} + 2\bar{q}_{12} \circ \underbrace{\bar{q}_{01} \circ q_{01}}_{Id} \circ \dot{q}_{12} \\
&= \bar{q}_{12} \circ \underbrace{(2\bar{q}_{01} \circ \dot{q}_{01})}_{\omega_{01}^1} \circ q_{12} + \underbrace{2\bar{q}_{12} \circ \dot{q}_{12}}_{\omega_{12}^2} \\
&= \bar{q}_{12} \circ \omega_{01}^1 \circ q_{12} + \omega_{12}^2 \\
&= \omega_{01}^2 + \omega_{12}^2.
\end{aligned}
$$

That is, we can add consecutive rotation speeds if they are expressed in the same referential.
In the case of the Cubsat, $\vec{\omega}_{02}^2$ is the satellite's rotation velocity $\vec{\omega}'$ expressed in body coordinates in the inertial referential model; we will note it $\vec{\omega}'_{Inertial}$ here. On the other hand, $\vec{\omega}_{12}^2$ is the satellite's rotation velocity $\vec{\omega}'$ expressed in body coordinates in the non-inertial referential model (i.e. in orbital reference frame, ORF); we will note it $\vec{\omega}'_{NonInertial}$.
$\vec{\omega}_{01}^1$ is the ORF rotation velocity expressed in the ORF, that is $\vec{\omega}_o$, while $\vec{\omega}_{01}^2$ is the same vector, transformed in the body referential. This transformation is performed by R^T from the non-inertial model ($\bar{q}_{12}$ in the above developement). In other words, we can link the $\vec{\omega}'$ vector from both inertial and non-inertial formulations (models) with

$$\vec{\omega}'_{Inertial} = R_{NonInertial}^T \vec{\omega}_o + \vec{\omega}'_{NonInertial}. \tag{84}$$

This is the speed to be used in computing the kinetic energy for the non-inertial model.

D Euler Angles to Quaternions

Three rotations by the Euler angles around each axis can be written as

$$R_\psi = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_\theta = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Combined together, they define the rotation matrix

$$R = R_\phi R_\theta R_\psi.$$

Those three rotations can also be expressed as quaternion rotations

$$\mathbf{q}_\phi = \begin{bmatrix} \cos\left(\frac{1}{2}\phi\right) \\ \sin\left(\frac{1}{2}\phi\right) \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{q}_\theta = \begin{bmatrix} \cos\left(\frac{1}{2}\theta\right) \\ 0 \\ \sin\left(\frac{1}{2}\theta\right) \\ 0 \end{bmatrix} \qquad \mathbf{q}_\psi = \begin{bmatrix} \cos\left(\frac{1}{2}\psi\right) \\ 0 \\ 0 \\ \sin\left(\frac{1}{2}\psi\right) \end{bmatrix}.$$

The resulting quaternion can then be obtained by multiplying those three together

$$\mathbf{q} = \mathbf{q}_\phi \circ \mathbf{q}_\theta \circ \mathbf{q}_\psi = \begin{bmatrix} \cos\left(\frac{1}{2}\phi\right)\cos\left(\frac{1}{2}\theta\right)\cos\left(\frac{1}{2}\psi\right) - \sin\left(\frac{1}{2}\phi\right)\sin\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\psi\right) \\ \cos\left(\frac{1}{2}\psi\right)\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\phi\right) + \cos\left(\frac{1}{2}\phi\right)\sin\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\psi\right) \\ \cos\left(\frac{1}{2}\psi\right)\cos\left(\frac{1}{2}\phi\right)\sin\left(\frac{1}{2}\theta\right) - \cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\phi\right)\sin\left(\frac{1}{2}\psi\right) \\ \cos\left(\frac{1}{2}\phi\right)\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\psi\right) + \cos\left(\frac{1}{2}\psi\right)\sin\left(\frac{1}{2}\phi\right)\sin\left(\frac{1}{2}\theta\right) \end{bmatrix}.$$

Note that this result depends on the convention used in the order and choice of the Euler angles and rotation axes! [2]

[2] Ref: *http://www.mathworks.com/access/helpdesk/help/toolbox/aeroblks/index.html?/access/helpdesk/help/toolbox/aeroblks/euleranglestoquaternions.html*

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